Doklady Mathematical Sciences 66 4/2

Instability of Eigenvalues Embedded in the Waveguide's Continuous Spectrum with Respect to Perturbations of Its Filling

A. N. Bogolyubov, M. D. Malykh, and A. G. Sveshnikov

Faculty of Physics, Moscow State University, Leninskie gory, Moscow, 119992 Russia
e-mail: malykham@mtu-net.ru
Presented by Academician V.A. Il'in April 3, 2002
Received April 8, 2002

We consider eigenvalues embedded in the continuous spectrum of the eigenvalue problem for a filled waveguide. A criterion for the existence of an infinite sequence of eigenvalues is stated for insertion-type fillings. The eigenvalues embedded in the continuous spectrum are shown to disappear under a small real perturbation of the filling.

Although there are examples of waveguide systems possessing eigenvalues embedded in a continuous spectrum, the necessary conditions for the emergence of embedded modes remain unclear. As noted in [1], it is thus reasonable to examine whether or not these modes persist under small perturbations of parameters in waveguide systems possessing trapped modes.

In a waveguide $\Omega = \left\{x \in \mathbb{R}^1, y \in S\right\}$, with the section S being a simply connected finite region in $\mathbb{R}^1$ or $\mathbb{R}^2$, we consider the eigenvalue problem

$$\begin{cases} \Delta u + eq(x,y)u = 0 \quad (x,y) \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u \in \overset{o}{W}{}_2^1(\Omega). \end{cases} \tag{1}$$

Here, $q(x, y)$ characterizes the filling of the waveguide. We assume that the waveguide filling is locally irregular; i.e., $q(x, y)$ is a piecewise continuous function and $\mathrm{Supp}\,[q(x, y) - 1] \equiv \Omega'$ is a finite region. It is also assumed that there is no damping in the waveguide; i.e., q is a real function.

A widespread practical case is a hollow waveguide Ω filled with a homogeneous substance with $q = 1$ and containing one or several plates with different $q \neq 1$ located perpendicularly to the waveguide axis; i.e., $q(x, y)$ is a piecewise continuous function of the single variable x. In this case, we can prove the existence of an infinite sequence of eigenvalues in problem (1). More exactly, the following result is valid.

Theorem 1. Let α_n^2 be the eigenvalues of the Dirichlet problem on S and ψ_n be the corresponding eigenfunctions. If $1 \leq q_0(x) \leq Q$, then, for any $n = 1, 2, \ldots$, the problem

$$\begin{cases} \Delta u + eq_0(x)u = 0 \quad (x, y) \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u \in \overset{o}{W}{}_2^1(\Omega) \end{cases} \tag{2}$$

has an eigenvalue $e^{(n)}$ on the interval $(\frac{\alpha_n^2}{Q},\ \alpha_n^2)$ that is associated with an eigenfunction of the form $u_n(x)\psi_n(y)$

Remark. The existence of an infinite sequence of eigenvalues for a waveguide with an insertion-type filling was indicated in [2]. In [3], the above estimates were obtained, and eigenvalues were calculated for various insertion-type fillings.

This theorem means, in particular, that for sufficiently small $q - 1$, even the eigenvalue $e^{(2)}$ of problem (2) is greater than α_1^2; therefore, the waveguide has an eigenfunction of the form $u_0(x, y) = u_2(x)\psi_2(y)$ associated with the eigenvalue $e^{(2)} > \alpha_1^2$, i.e., an eigenvalue embedded in a continuous spectrum. Let us now find out whether or not this eigenvalue is preserved if the filling is perturbed so that

$$q(x, y) = q_0(x) + \varepsilon q_1(x, y),$$

where q_1 is a real function and ε characterizes the smallness of the perturbation.

It was shown in [4] that no more than one eigenvalue $e(\varepsilon)$ of the perturbed problem (1) exists in a sufficiently small neighborhood of a simple eigenvalue e_0 of the unperturbed problem (2). Moreover, if the former eigenvalue exists, then it and the corresponding eigenfunction $u(x, y; \varepsilon)$ are analytic functions of ε and are regular at zero.

To prove this statement, we use the resolvent of the regular waveguide. Its explicit expression is given by

$$R_0(e)v = \sum_{n=1}^{\infty} \frac{i}{2\sqrt{e-\alpha_n^2}} \int_{\Omega} d\xi d\eta e^{i\sqrt{e-\alpha_n^2}|x-\xi|} \psi_n(\eta)\psi_n(y)\, v(\xi,\eta)$$

and it maps $L^2(\Omega')$ to $\overset{o}{W}{}^1_{2,\,\mathrm{loc}}(\Omega)$. The substitution $u = R_0(e)v$ made in (1) gives an integral equation for v:

$$v - \mathfrak{A}(e,\varepsilon)v = 0, \quad \text{where} \quad \mathfrak{A}(e,\varepsilon) = -e(q(x,y;\varepsilon)-1)R_0(e). \tag{3}$$

Since $\operatorname{Supp} q - 1$ is bounded, $\mathfrak{A}(e,\varepsilon)$ is a compact operator function holomorphic on a Riemann surface $\mathfrak{f}$ with branch points at α_n^2. This procedure for reducing the original problem to an integral equation is a modification of the procedure suggested in [5]; however, the former is easier to justify for weak solutions.

It was shown in [4] that the set of eigenvalues of $\mathfrak{A}(e,\varepsilon)$ that lie on the principal sheet (where all roots $\sqrt{e-\alpha_n^2}$ have principal values) coincides with the set of all eigenvalues of problem (1). If $\mathfrak{A}$ has a simple eigenvalue e_0 for $q = q_0$, then, in a sufficiently small neighborhood of that eigenvalue, there exists a unique eigenvalue $e(\varepsilon)$ depending analytically on ε.

If e_0 is an isolated simple eigenvalue of problem (2), then e_0 is a simple eigenvalue of $\mathfrak{A}(e,0)$ lying inside the principal sheet. Consequently, a unique eigenvalue $e(\varepsilon)$ of $\mathfrak{A}(e,\varepsilon)$ lies in a sufficiently small neighborhood of e_0. Since the small neighborhood of e_0 lies on the principal sheet, $e(\varepsilon)$ is a unique perturbed eigenvalue of problem (1) with a perturbed filling, and it tends to e_0 as $\varepsilon \longrightarrow 0$.

However, if e_0 is an eigenvalue embedded in the continuous spectrum of problem (2), then e_0 is an eigenvalue of $\mathfrak{A}(e,0)$ lying on the boundary of the principal sheet. Therefore, although a single eigenvalue $e(\varepsilon)$ of $\mathfrak{A}(e,\varepsilon)$ lies in a sufficiently small neighborhood of e_0, this eigenvalue may not lie on the principal sheet and, hence, may not be an eigenvalue of problem (1) with a perturbed filling. This means that there exists no more than one eigenvalue of (1) that tends to e_0 as $\varepsilon \longrightarrow 0$. Moreover, if such an eigenvalue exists, it coincides with the corresponding eigenvalue of $\mathfrak{A}(e,\varepsilon)$ and, hence, depends analytically on ε, as stated above.

Now, we assume that an eigenvalue of the perturbed problem (1) exists in a neighborhood of e_0 $e^{(2)}$ for any $q_1(x,y)$. Then, this eigenvalue and the corresponding eigenfunction can be represented as series expansions:

$$e(\varepsilon) = e_0 + e_1\varepsilon + \dots, \quad u(x,y;\varepsilon) = u_2(x)\psi_2(y) + \varepsilon u_1(x,y) + \dots .$$

Multiplying (1) by $\psi_1(y)$ and integrating the result over the entire section S, we obtain

$$\frac{d^2}{dx^2}\int\limits_S dyu(x,y)\psi_1(y) + e\int\limits_S dy\, q(x,y)u(x,y)\psi_1(y) = \alpha_1^2\int\limits_S dyu(x,y)\psi_1(y).$$

Substituting the series expansions of $e(\varepsilon)$ and $u(\varepsilon)$ into this equation and introducing

$$\int\limits_S dyu(x,y)_1\psi_1(y) = u_{1,1}(x),$$

we obtain, up to the first perturbation order,

$$\frac{d^2u_{1,1}}{dx^2} + [e_0q_0(x) - \alpha_1^2]u_{1,1} = e_0u_2(x)\int\limits_S dy\, q_1(x,y)\psi_1(y)\psi_2(y).$$

For $u(x,y;\varepsilon)$ to belong to L^2, it is necessary that $u_{1,1}(x)$ be in $L^2(\mathbb{R}^1)$. Since the support of the perturbed filling $q(x,y)-1$ is bounded, this equation has a solution in L^2 only under rather special conditions on $q_1(x,y)$. Thus, we have proved the following statement (cf. [6]).

Theorem 2. There exist piecewise continuous real perturbations $q_1(x, y)$ of the initial filling $q_0(x)$ such that there are no perturbed eigenvalues in the neighborhood of the unperturbed eigenvalue.

Moreover, it can be seen from the proof that the eigenvalue $e_0 = e^{(2)}$ corresponding to the eigenfunction $u_0(x, y) = u_2(x)\psi_2(y)$ is stable only with respect to those perturbations for which the equation

$$\frac{d^2 w}{dx^2} + [e_0 q_0(x) - \alpha_1^2]w = e_0 u_2(x) \int\limits_S dy\, q_1(x, y)\psi_1(y)\psi_2(y) \tag{4}$$

has a solution in $L^2(\mathbb{R}^1)$.

The simplest example illustrating this statement is the case where

$$\Omega = \{x \in \mathbb{R}^1, \quad y \in [0,\ +\pi]\}, \quad \Omega' = \{x \in [-1, 1], \quad y \in [0,\ +\pi]\}$$

and

$$q_0\,(x) = \begin{cases} q_0, & x \in (-1, +1) \\ 1, & \text{otherwise} \end{cases}.$$

It is easy to show that the smallest eigenvalue corresponding to eigenfunctions of the form $u_2(x)\psi_2(x)$ then disappears under a perturbation of the form

$$q_1(x, y) = \begin{cases} \frac{\psi_2(y)}{\psi_1(y)} \sin \sqrt{e_0 q_0 - \alpha_1^2}(x \pm 1) \cos \sqrt{\alpha_2^2 - e_0 q_0}\, x, & |x| < 1 \\ 1, \quad \text{otherwise} \end{cases} \tag{5}$$

if this eigenvalue is embedded in a continuous spectrum.

The basic meaning of the theorem proved is that eigenvalues embedded in a continuous spectrum are unstable with respect to small perturbations of the waveguide filling. This property is rather unexpected, because an eigenvalue usually disappears only under a complex-valued perturbation of the filling, i.e., after the introduction of damping.

Acknowledgments. This work was supported by the Russian Foundation for Basic Research (project nos. 02-01-00271, 00-01-00111) and the program "Universities of Russia" (project no. UR.03.02.010).

References

1. Davies, E.B. and Parnovski, L., Quart. J. Mech. Appl. Math., 1998, vol. 51, pp. 477–492.

2. Delitsyn, A.L., Vestn. Mosk. Univ., Ser. 3: Fiz., Astron., 2001, no. 3, pp. 75–76.

3. Bogolyubov, A.N., Delitsyn, A.L., and Malykh, M.D., Vestn. Mosk. Univ., Ser. 3: Fiz., Astron., 2001, no. 5, pp. 23–25.

4. Bogolyubov, A.N. and Malykh, M.D., Zh. Radioelektron., 2002, no. 2.

5. Shestopalov, V.P., Spektral'naya teoriya i vozbuzhdenie otkrytykh struktur (Spectral Theory and Excitation of Open Structures), Moscow: Nauka, 1987.

6. Malykh, M.D., Vestn. Mosk. Univ., Ser. 3: Fiz., Astron., 2002, no. 1, pp. 61–62.