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ON TRANSCENDENTAL FUNCTIONS ARISING FROM INTEGRATING DIFFERENTIAL EQUATIONS IN FINITE TERMS

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In this paper, we discuss a version of Galois theory for systems of ordinary differential equations in which there is no fixed list of allowed transcendental operations. We prove a theorem saying that the field of integrals of a system of differential equations is equivalent to the field of rational functions on a hypersurface having a continuous group of birational automorphisms whose dimension coincides with the number of algebraically independent transcendentals introduced by integrating the system.

The suggested construction is a development of the algebraic ideas presented by Paul Painlevé in his Stockholm lectures. Bibliography: 34 titles.

1. Introduction

The idea of solvability in finite terms is one of those obscure notions that are commonly understood to be a historical convention and are in essence the subject of a general agreement on what this term actually means (see [1]). Thus, when dealing with geometric construction problems, one fixes a set of tools and a list of rules for using them; when studying algebraic equations in the framework of Galois theory, one tries to express a solution in terms of a few radicals; in the theory of differential equations, one tries to express a general solution or integral in terms of Liouville functions, which means that not only algebraic operations are allowed, but also calculations of quadratures and exponents, and between the existence of a Liouville integral and a solution there is a very nontrivial connection [2, 3].

In practice, when solving nonlinear differential equations with the help of computer algebra systems, one uses different methods. The first computer solver of differential equations, written by Moses [4] in the early 1960s, was based on the following simple observation: one can decide whether a given differential equation

$$p(x,y)dx + q(x,y)dy = 0$$

has an integrating factor of the form $\mu(x)$ or $\mu(y)$ in finitely many steps. It is important to mention that if a system has a factor of this form, then it is a Liouville function of the variable. In *Maple*, integration of differential equations uses special classes of groups for which one can decide in finitely many steps whether a given differential equation has a group of symmetries of this class and compute its infinitesimal operator [5–8]. The power of *Maple* in solving first-order differential equations is due to the fact that 78% of examples from the Kamke handbook have linear groups of symmetries [8]. In this case, the infinitesimal operator can again be expressed in terms of Liouville functions, although this is not assumed a priori.

It is noteworthy that the set of tools used for geometric constructions and the rules of using them have been changing with time [9, Remark 33]. The list of transcendental functions that one frequently uses, on the contrary, was formed in the time of Gauss and since then has not undergone any significant changes. When using, say, Painlevé transcendents, we feel that we go over some line. Can we actually find a feature that describes the commonly used functions as a mathematical rather then sociocultural phenomenon? In other words, can we construct

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a version of differential Galois theory for nonlinear differential equations in which the list of admissible operations is not postulated from the beginning?

All commonly used functions are meromorphic solutions of differential equations. Lazarus Fuchs suggested to find all first-order differential equations whose general solution does not have movable singularities except, maybe, poles; now this property is called the Painlevé property, [10–12]. First-order equations having this property either can be reduced to a linear second-order equation or can be solved in elementary or elliptic functions, [13, Chap. 2]. However, Painlevé ([13, Chap. 3]) discovered that among the second-order differential equations having the property that bears his name there are equations that lead to absolutely new transcendental functions, so that this property cannot be used to describe the class of commonly used functions.

In the early 20th century, such a radical expansion of the class of commonly used functions was received with great enthusiasm, which explains the choice of topics for the 6th Lobachevsky prize contest held in 1912, see [15]. This perhaps also explains the lack of interest to other, purely algebraic, ideas expressed in the famous Stockholm lectures by Painlevé [16]. He noticed that general solutions of differential equations solvable in commonly used functions are not only meromorphic functions of the independent variable, but also algebraic functions of the constants. Painlevé managed to invert this assertion for first- and second-order equations. Briefly, the assertion he proved can be stated as follows: if the general solution depends algebraically on the constants of integration, then integrating this equation does not lead out of the realm of commonly used functions complemented by the Abel functions.

In more detail: a first-order equation whose general solution depends algebraically on an appropriate constant can be reduced by an algebraic change to a first-order equation satisfying the Painlevé property. We do not cite the original assertions, since in both the article [17] and the original Lectures they are stated in too much generality, the mistake corrected much later in the appendix written by Painlevé for Boutroux's essay [18]. The study of second-order equations in the Lectures is divided into two parts. First, it is proved that every such equation can be reduced by an algebraic change to an equation whose general solution depends rationally on the initial conditions [16, p. 242]. Second, the general solution of such an equation is described in detail. We give an abridged version of the statement.

Proposition 1 (Painlevé, [16, p. 381])**.** *If the general solution y of a given second-order equation*

$$f(x, \dot{x}, \ddot{x}; t) = 0$$

(where f is a polynomial in $x, \dot{x}, \ddot{x}$) depends rationally on the constants $x_0, \dot{x}_0, \ddot{x}_0$ related by

$$f(x_0, \dot{x}_0, \ddot{x}_0; t_0) = 0,$$

then this integral belongs to one of the following categories:

(1) *either this integral can be expressed algebraically;*

(2) *or x can be expressed rationally in terms of the elliptic functions $\wp(u+C)$ and $\wp'(u+C)$ where u can be expressed in terms of t by the following quadrature:*

$$u = \int h(t)dt,$$

i.e., $x = R(\wp(u+C), \wp'(u+C))$ and the coefficients of the function R can be expressed algebraically in terms of the coefficients of the original differential equation and the second integration constant;

(3) *or x can be expressed rationally in Abel functions of the form $\mathrm{Al}(u, v)$ and their derivatives with respect to u and v where u and v can be expressed in terms of t by the*

quadratures

$$u = \int h(t)dt + C_1, \quad v = \int k(t)dt + C_2;$$

(4) *or the general solution can be expressed rationally in* $y(t)$*, i.e.,* $x = R(y)$*, where* y *satisfies the Riccati equation*

$$\dot{y} = -y^2 + \gamma(t),$$

R and γ can be expressed algebraically in terms of the coefficients of the original differential equation and an arbitrary constant C;

(5) *or* $x = R(y, \wp(u + C), \wp'(u + C))$ *where u is given by the quadrature*

$$u = \int h(t)dt,$$

y satisfies the Riccati equation

$$\dot{y} = -y^2 + \gamma(t),$$

R can be expressed algebraically in terms of the coefficients of the original differential equation, and γ may depend rationally on $\wp(u + C)$ and $\wp'(u + C)$;

(6) *or the original equation can be reduced by an algebraic change to a linear differential equation.*

This theorem leads us to an unexpected conclusion: *by fixing the algebraic properties of the general solution one can derive a class of commonly used transcendental functions.* In the general case, one can pose the following problem.

Problem 1. *Describe the transcendental operations required for representing the solutions of a system of differential equations if it is known that the general solution of this system depends algebraically on the constants.*

The complexity of the statement of Proposition 1 makes us search for more convenient objects of study. Painlevé started from problems whose solutions were meromorphic functions of t, and hence chose to study solutions of the Cauchy problem

$$\begin{cases} f(x, \dot{x}, \ddot{x}; t) = 0, \\ (x, \dot{x}, \ddot{x})|_{t=t_0} = (x_0, \dot{x}_0, \ddot{x}_0), \end{cases}$$

which define a birational correspondence between the surfaces

$$f(x, y, z; t) = 0 \quad \text{and} \quad f(x, y, z; t_0) = 0.$$

These correspondences are hard to incorporate into Galois theory where the main objects of study are fields. Yet the integrals of a differential equation form fields, thus they are going to be our main objects in what follows.

The purpose of this paper is to give an outline of a Galois theory for differential equations in which one does not fix a set of admissible transcendental operations (for first-order differential equations, such a theory was outlined in our paper [19]). Let us briefly describe the theory, referring the reader to the subsequent sections for more precise definitions. If a system of differential equations

$$\begin{cases} f_1(x_1, \dots, \dot{x_1}, \dots; t) = 0, \\ f_2(x_1, \dots, \dot{x_1}, \dots; t) = 0, \\ \dots \\ f_n(x_1, \dots, \dot{x_1}, \dots; t) = 0 \end{cases}$$

has an integral depending on $x_1, \ldots, x_n$ algebraically, then it has an integral that is rational on the manifold V in the affine space A^{2n} given by the equations

$$\begin{cases} f_1(x_1, \ldots, x_{n+1}, \ldots; t) = 0, \\ f_2(x_1, \ldots, x_{n+1}, \ldots; t) = 0, \\ \ldots \\ f_n(x_1, \ldots, x_{n+1}, \ldots; t) = 0. \end{cases}$$

All rational integrals of a system of differential equations form a field, which will be called the field of integrals.

The coefficients of the original system of differential equations generate a differential field over the field of constants $\mathbb{C}$; we will call it the field of basic functions and assume that it is given together with the given equation. The coefficients of rational integrals are also some functions of t. If these coefficients belong to the field of basic functions, then the system has an algebraic integral, which can immediately be added to the system. Otherwise these coefficients generate a field over the field of basic functions, which will be called the field of transcendentals introduced by integrating.

Problem 2. *Describe the transcendental operations required for defining the field of transcendentals introduced by integrating a system of differential equations.*

The key to this problem is Theorem 11, according to which *the field of integrals of a system of differential equations is equivalent to the field of rational functions on a hypersurface having a continuous group of birational automorphisms whose dimension coincides with the number of algebraically independent transcendentals introduced by integrating the system.* The proof of this theorem is the main purpose of the subsequent sections.

Remark. At the scientific session of the National Research Nuclear University MEPhI-2015, Professor N. A. Kudryashov drew my attention to Hiroshi Umemura's works devoted to the nonelementarity of Painlevé transcendentals. In [20], Umemura gave a modern exposition of some elements of the above-mentioned works by Painlevé in the form of a peculiar Galois theory. The main object of his theories is a differential equations whose general solution depends rationally on the initial conditions.

2. Differential equations over the field of basic functions

The algebraic theory of differential equations originates in Weierstrass' lectures known in Leo Königsberger's presentation [21,22] and in Painlevé's works summarized in his Stockholm lectures [16]. Old authors worked by default with analytic functions over the field $\mathbb{C}$ and treated degenerate cases too loosely. Modern authors, on the contrary, usually consider solutions of differential equations as curves on differential manifolds using the standard topology of $\mathbb{R}^n$, see [23]. However, when studying algebraic questions, it is natural to work in the Zariski topology.

2.1. Fields of functions. Ordinary differential equations determine a connection between functions of an independent variable (say, time t) and their derivatives. By increasing the order of the system of differential equations under consideration, one can usually turn the right-hand side of the system into a polynomial in the functions and their derivatives whose coefficients are arbitrary analytic functions of the variable t. Let us assume that the coefficients of the equations under consideration belong to a given field of functions.

Definition 1. *A* field of functions *of a variable t is a differential field whose elements are meromorphic functions of the independent variable t in a simply connected domain of the* *complex plane; the differentiation of the field is the differentiation with respect to t, and the field of constants is the field $\mathbb{C}$ of complex numbers.*

A field of functions that contains the coefficients of the differential equations under consideration will be called the field of basic functions. *A set of functions of the variable t that are algebraically independent over the field of basic functions will be called transcendental functions over k, or just transcendentals.*

The above-mentioned range of the variable t is called the domain of the field of functions k and denoted by $\mathrm{Dom}(k)$. By the monodromy theorem, functions of the field k are uniquely determined in this domain.

The value of a function φ from the field k at a point $t = a$ is denoted by $\varphi|_{t=a}$. If g is a polynomial of the ring $k[x_1, \dots, x_n]$, then its coefficients depend on t; replacing t with a number $t = a$ from the field of constants $\mathbb{C}$, we get another element of this field, which we denote by $g|_{t=a}$. The differentiation with respect to t can be extended to this ring by setting

$$\frac{\partial}{\partial t}(ax_1^{m_1} \dots x_n^{m_n} + \dots) = \dot{a}x_1^{m_1} \dots x_n^{m_n} + \dots.$$

We keep brackets for another thing: if in a polynomial g of the ring $k[x_1, \dots, x_n]$ we replace the variables $x_1, \dots, x_n$ by a point q of the affine space A^n over the field k, we get an element of the field k, i.e., a function of the variable t, which we denote by $g(q)$. The differentiation of the field satisfies the Leibniz rule, hence

$$\frac{dg(q)}{dt} = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(q) \cdot \dot{q}_i + \frac{\partial g}{\partial t}(q).$$

2.2. Differential equations. In general, the field of basic functions k is closed with respect to neither algebraic nor differential equations. Let us embed it into an algebraically closed field K. Unless otherwise stated, we assume that the field of constants of the field K coincides with $\mathbb{C}$. In this case, the affine space A^n over K, but not over k, is a standard object of algebraic geometry, [24, Chap. 1].

Remark 2.1. No algebraically closed extension of the field of basic functions can be regarded as a field of functions in the sense of Definition 1, since any point of the domain of the field of basic functions is a branch point for some element of the extension, and thus one cannot indicate a domain in which the elements of the extension are one-valued functions. In particular, the substitution $t = a$ cannot be regarded as a uniquely defined map from the extension K onto $\mathbb{C}$.

The affine space A^n_k will be regarded as a subset of the affine space A^n_K. If $\mathfrak{a}$ is an ideal of the ring $k[x_1, \dots, x_n]$, then the set $K[x_1, \dots, x_n]\mathfrak{a}$ is an ideal of the ring $K[x_1, \dots, x_n]$, which is called the completion of the ideal $\mathfrak{a}$ in the field K and denoted by $\mathfrak{a}^K$. The set of points of the affine space A^n over k or K at which all polynomials from $\mathfrak{a}$ vanish is denoted by $V(\mathfrak{a}/k)$ or $V(\mathfrak{a}/K)$, respectively.

In our case, the fields k and K have an additional structure: the differentiation. It can be applied to every coordinate of a point q of the affine space A^n, and thus we obtain n elements of this field. With a point q of the affine space A^n over K we associate the point

$$\dot{q} = (q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$$

of the affine space A^{2n} over K. Here the relation $\dot{q} \in V(\mathfrak{a}/K)$ means that

$$\begin{cases} g_1(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) = 0, \\ \dots, \\ g_m(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) = 0. \end{cases}$$

Definition 2. *A solution in a differential extension K of the field of basic functions k of a system of differential equations*

$$\begin{cases} g_1(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) = 0, \\ \cdots, \\ g_m(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) = 0 \end{cases} \tag{1}$$

whose left-hand sides generate an ideal $\mathfrak{a} = (g_1, \dots, g_m)$ of the ring $k[x_1, \dots, x_{2n}]$ is a point q of the affine space A^n over K such that $\dot{q}$ belongs to $V(\mathfrak{a}/K)$.

The set of solutions depends only on the ideal $\mathfrak{a}$, but not on the choice of its generators; we denote this set by $S(\mathfrak{a}/K)$.

Remark 2.2. We do not assume that the number of differential equations coincides with the number of unknowns, since otherwise we would have to exclude from consideration very important examples related to mechanics. For example, in the three-body problem, accelerations can be expressed rationally in terms of the coordinates of the bodies and the distances between them, thus taking the Cartesian coordinates of the bodies, their velocities, and the distances between them as variables, we get a system of differential equations of the form (1) if we add to the Newton equations the algebraic equations relating the Cartesian coordinates of the bodies to the distances between them. It is more natural to consider the Newton equations for such a problem on a manifold, yet this is not a vital necessity.

Remark 2.3. In the theory based on the topology of $\mathbb{R}^n$, solutions of differential equations are curves on differential manifolds tangents to which nullify the Cartan distribution [23]; in the Zariski topology, the situation is even simpler: a solution is merely a point of an affine space regarded over a field of functions.

2.3. Totally consistent and closed systems of differential equations. In Definition 2 we did not fix the number of differential equations, there can be too many of them for the set of solutions to be nonempty or too few for the derivatives $\dot{x}_i$ to be expressible in terms of the functions x_j.

A condition showing that an extension K has sufficiently many solutions of (1) is formulated analogously to Hilbert's Nullstellensatz [25].

Definition 3. *A system of differential equations* (1) *is totally consistent over a field K if for any polynomial f from $k[x_1, \dots, x_{2n}]$ vanishing on every solution from $S(\mathfrak{a}/K)$, one can find an integer r such that f^r belongs to the ideal $\mathfrak{a}$ generated by the right-hand sides of the equations of the system.*

A condition showing that the number of equations is not too small will be formulated only for irreducible systems.

Definition 4. *A system* (1) *is irreducible in a field K if the topological space $V(\mathfrak{a}/K)$ is irreducible.*

Without introducing new transcendental functions one can extend the fields of basic functions in such a way that the given system of differential equations splits into several systems whose left-hand sides generate irreducible manifolds in the affine space over any extension of the field of basic functions. This allows us to consider only irreducible systems of differential equations in what follows. By an irreducible system of differential equations without any specified field we mean a system that is irreducible over the algebraic closure of the field of basic functions.

Thus consider an irreducible system of differential equations (1) whose left-hand sides generate an ideal $\mathfrak{p}$, whose complement $\mathfrak{p}^K$ is, consequently, prime. Elements of the field of fractions of the ring $K[x_1,\dots,x_{2n}]/\mathfrak{p}^K$, which we denote by $R(\mathfrak{p}/K)$, will be interpreted as maps from an open subset of the affine space $V(\mathfrak{p}/K)$ to K and called rational functions on the affine space $V(\mathfrak{p}/K)$. Elements of the field of fractions of the ring $k[x_1,\dots,x_{2n}]/\mathfrak{p}$, which we denote by $R(\mathfrak{p}/k)$, can be regarded not only as rational functions on $V(\mathfrak{p}/k)$ with values in k, but also as rational functions on $V(\mathfrak{p}/K)$ with values in K, i.e., this field of fractions can be regarded as a subfield in $R(\mathfrak{p}/K)$.

Definition 5. *An irreducible system of differential equations over the field k whose right-hand sides generate an ideal $\mathfrak{p}$ of the ring $k[x_1,\dots,x_{2n}]$ is closed if there exists a differentiation D of the field of rational functions on $V(\mathfrak{p}/k)$ such that for any extension K of the field k the equality*

$$\frac{df(q)}{dt} = Df(q)$$

holds for all $q \in S(\mathfrak{p}/K)$ and $f \in k[x_1,\dots,x_{2n}]$.

Such a differentiation D can be extended to $R(\mathfrak{p}/K)$ for any K, and for any monomial we have

$$Dx_1^{n_1}\dots x_{2n}^{n_{2n}} \in R(\mathfrak{p}/k).$$

3. Puiseux extensions

We turn to the question on the existence of an extension of the original field over which an irreducible system of differential equations is totally consistent and closed.

3.1. Fields of Puiseux series. In the theory of algebraic numbers, one can embed all fields into the field $\mathbb{C}$ in which all algebraic equations have roots. An analog of the so-called fundamental theorem of algebra in the theory of differential equations is the Cauchy theorem, which gives solutions of systems of differential equations in the form of power series. For an extension of the field of basic functions to be algebraically closed, we should also take into account fractional powers. Formal series of the form

$$a_0t^{n_0} + a_1t^{n_1} + \dots,$$

where $n_0 < n_1 < \dots$ are rational numbers with a common denominator and $a_0, a_1, \dots$ are elements of a field $\mathbb{K}$, are called Puiseux series in powers of t with coefficients in the field $\mathbb{K}$; if $\mathbb{K}$ is algebraically closed of characteristic 0, then the set of such series is an algebraically closed field, [26, Theorem 2.1.5].

Definition 6. *The Puiseux extension of the field of basic functions k at a point $t = a$ of the domain $\mathrm{Dom}(k)$ is the field of Puiseux series in powers of $t - a$ with coefficients in the field of constants of the field of basic functions (i.e., by default, the field $\mathbb{C}$); we denote it by $P_a(k)$. Instead of $V(\mathfrak{a}/P_a(k))$ and $S(\mathfrak{a}/P_a(k))$, for brevity we write $V_a(\mathfrak{a})$ and $S_a(\mathfrak{a})$.*

Since every function from the field of basic functions can be expanded in $\mathrm{Dom}(k)$ into a Laurent series with finitely many negative powers, a Puiseux extension of the field of basic functions is an algebraically closed differential extension of the field of basic functions. It is not, however, a field of functions in the sense of Definition 1, since Definition 6 contains assumptions neither on convergence nor on the existence of single-valued analytic branches.

3.2. The consistency of a system of differential equations. Instead of the general Definition 3, we consider first the following one.

Definition 7. *A system of differential equations* (1) *is totally consistent if for a polynomial* f *from* $k[x_1, \dots, x_{2n}]$ *that vanishes on every solution in the Puiseux extension* $P_a(k)$ *and for every* a *from the domain of the field of basic functions* k*, one can find an integer* r *such that* f^r *belongs to the ideal* $\mathfrak{a}$ *generated by the right-hand sides of the equations of this system.*

Let us show that an irreducible system of n differential equations in n unknown functions

$$\begin{cases} g_1(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) = 0, \\ \cdots \\ g_n(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) = 0 \end{cases} \tag{2}$$

is totally consistent in this sense.

Theorem 1. *Assume that the Jacobian*

$$j = \frac{\partial g_1, \dots, g_n}{\partial x_{n+1}, \dots, x_{2n}}$$

of an irreducible system (2) *does not belong to the ideal* $\mathfrak{p} = (g_1, \dots, g_n)$ *generated by the left-hand sides of the equations of the system. If a polynomial* f *from* $k[x_1, \dots, x_{2n}]$ *vanishes on every solution from* $S_a(\mathfrak{p})$ *for every* a *from the domain of the field of basic functions* k*, then* f *belongs to the ideal* $\mathfrak{p}$*.*

Proof. The ideal $\mathfrak{a} = (j, g_1, \dots, g_n)$ determines a closed subset $V_a(\mathfrak{a})$ on the manifold $V_a(\mathfrak{p})$. Denote by U the complement to $V_a(\mathfrak{a})$ and take a point p in this domain, which is actually $2n$ Puiseux series in powers of $t - a$.

First assume that these series converge in some neighborhood of the point $t = a$. A point p of the manifold $V(\mathfrak{p})$ belongs to U if the series $j(p)$ in powers of $t - a$ does not vanish identically, i.e., if there exists a point $t = b$ such that $p|_{t=b}$ is a point of the affine space A^{2n} over $\mathbb{C}$ and $j(p)|_{t=b} \neq 0$.

When considering the Cauchy problem, it is convenient, following Painlevé, to introduce a new independent variable $\bar{t}$ and agree that the bar over an expression means replacing t with $\bar{t}$. Denote by L the field of Laurent series in powers of $t - b$ with coefficients in the field $\bar{k}$. Then the Cauchy theorem says that for t and $\bar{t}$ sufficiently close to b, the system of differential equations under consideration has a solution q in the field L which for $t = \bar{t}$ takes the value $\dot{q} = \bar{p}$. Making $\bar{t}$ take arbitrary complex values close to $t = b$, we get solutions in the field $P_b(k)$. By the assumption, the polynomial f vanishes on any solutions from $P_b(k)$ and, in particular, on the series $\dot{q}$ whatever value of $\bar{t}$ we substitute, and hence on these series regarded as elements of the field L. Substituting into

$$f(\dot{q}) = 0$$

the value $t = \bar{t}$, we get

$$\overline{f}(\overline{p}) = 0;$$

since $\bar{t}$, as well as t, is a dependent variable, this only means that $f(p) = 0$.

Now let p' be an element of the domain U, which is given by formal Puiseux series. For it we can find a Puiseux series p'' converging in a neighborhood of $t = a$ such that any number of the first coefficients of the series $f(p')$ and $f(p'')$ coincide. Since the polynomial f vanishes on p'', all coefficients of the series $f(p')$ equal to zero, i.e., f vanishes on the whole open set U.

By our assumption, the extension of the ideal $\mathfrak{p}$ is a prime ideal, thus the space $V_a(\mathfrak{p})$ is irreducible and, therefore, the equality $f(p) = 0$ can be extended to the whole space $V_a(\mathfrak{p})$. Since a Puiseux extension gives an algebraically closed field, by Hilbert's Nullstellensatz f belongs to the extension of the ideal $\mathfrak{p}$. By the assumption, the coefficients of f lie in the field of basic functions, hence $f \in \mathfrak{p}$. □

Remark 3.1. For the sake of the construction used in the proof of this theorem, above it was allowed to consider extensions of the field k with fields of constants larger than $\mathbb{C}$.

It would be much more convenient to use one extension instead of the whole family of Puiseux extensions, assuming, in full agreement with Definition 3, the following.

Definition 8. *A system of differential equations* (1) *is called totally consistent over an extension K of the field of basic functions if for a polynomial f from $K[x_1, \dots, x_{2n}]$ that vanishes on any solution in the field K, one can find an integer r such that f^r belongs to the ideal $\mathfrak{a}^K$ generated by the right-hand sides of the equations of this system.*

By the Painlevé theorem, from the domain of the field of basic functions one can remove a set of points, called fixed singularities, so that in the remaining domain, analytic solutions of the given first-order equation

$$g(x, \dot{x}) = 0, \quad g \in k[x_1, x_2],$$

can be continued without meeting any other singularities apart from algebraic ones, [13, p. 51].

Theorem 2. *An irreducible first-order differential equation*

$$g(x, \dot{x}) = 0, \quad g \in k[x_1, x_2],$$

is totally consistent over the field $P_a(k)$ where $t = a$ is any point of the domain of the field of basic functions different from the fixed singularities of this equation.

The proof of this theorem repeats the proof of the previous theorem word by word, except that now $f(p)$ vanishes not because the series q belongs to $S_b(\mathfrak{p})$ for any fixed $\bar{t}$, but because we can continue it to the point $t = a$ and get a series from $S_a(\mathfrak{p})$.

Unfortunately, already for known mechanical systems, the algebraicity of movable singularities must be proved separately. For example, for the three-body problem it was established along the real axis and for real initial data that do not nullify the angular momentum of the system, [27]. For systems solved with respect to derivatives, this property was established under the assumption that the coefficients are not subject to special conditions, [16].

The proofs given below in Sec. 4 look simpler for systems totally consistent over the field K, but without extra effort can be extended to systems totally consistent in the sense of Definition 7.

3.3. The closeness of the differential equations of a system. Under the conditions of Theorem 1, it is not difficult to prove the closeness of a system (2) of n differential equations in n unknown functions in the sense of Definition 5.

Let q be any solution from $S(\mathfrak{p}/K)$; then

$$f_i(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) = 0,$$

and thus

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(q) \cdot q_{i+n} + \sum_{j=n+1}^{2n} \frac{\partial f_i}{\partial x_j}(q) \cdot \dot{q}_i + \frac{\partial f_i}{\partial t}(q) = 0.$$

If the Jacobian of the system (2) does not belong to the ideal $\mathfrak{p}$, then the system

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \cdot x_{i+n} + \sum_{j=n+1}^{2n} \frac{\partial f_i}{\partial x_j} \cdot r_i + \frac{\partial f_i}{\partial t} = 0, \quad i = 1, \dots, n \, , \end{cases}$$

has a solution $(r_1, \dots, r_n)$ in the field of rational functions $R(\mathfrak{p}/k)$ over the field K. Let us extend the differentiation with respect to t of the field K to a differentiation D of the field $R(\mathfrak{p}/K)$ by assuming

$$Dx_i = x_{i+n}, \quad Dx_{i+n} = r_i, \quad i = 1, \dots, n.$$

Then in any point q of $S(\mathfrak{p}/K)$,

$$Dx_i(q) = x_{i+n}(q) = q_{i+n} = \dot{q}_i, \quad i = 1, \dots, n,$$

and since the system

$$\left\{ \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(q) \cdot x_{i+n}(q) + \sum_{j=n+1}^{2n} \frac{\partial f_i}{\partial x_j}(q) \cdot z_i + \frac{\partial f_i}{\partial t}(q) = 0, \quad i = 1, \dots, n, \right.$$

has in K a unique solution $(z_1, \dots, z_n)$, we have

$$Dx_{n+i}(q) = z_i = \dot{q}_{n+i}, \quad i = 1, \dots, n.$$

From the $2n$ equations

$$Dx_i(q) = \dot{q}_i, \quad i = 1, \dots, 2n,$$

and the Leibniz rule it follows that

$$\frac{df(q)}{dt} = Df(q).$$

Thus we have proved the following theorem.

Theorem 3. *If the Jacobian*

$$j = \frac{\partial g_1, \dots, g_n}{\partial x_{n+1}, \dots, x_{2n}}$$

of an irreducible system (2) *does not belong to the ideal $\mathfrak{p} = (g_1, \dots, g_n)$ generated by the left-hand sides of the equations of this system, then this system is closed.*

4. Rational integrals of a system of differential equations

Let us turn to integrals of an irreducible closed system (1) totally consistent over K whose right-hand sides generate an ideal $\mathfrak{p}$.

4.1. Algebraic integrals. In the theory of differential equations, and especially in mechanics, by an algebraic integral of motion one means an algebraic function of the variables $x_1, \dots, x_n$ that is constant on any solution.

Definition 9. *We say that an equation*

$$a_0 z^s + \cdots + a_s = 0, \quad a_i \in K[x_1, \dots, x_n],$$

determines an algebraic integral of motion if on any solution q from $S(\mathfrak{a}/K)$ the equation

$$a_0(q) z^n + \cdots + a_n(q) = 0$$

has a root in the field of constants of the field K. This root will be called a fixed root.

A nonconstant function f rational on $V(\mathfrak{p}/K)$ will be called a rational integral of motion if on any solution $q \in S(\mathfrak{p}/K)$ the expression $f(\dot{q})$ lies in the field of constants.

Theorem 4. *If an irreducible system is totally consistent and closed over an extension K of the field of basic functions k, then any equation*

$$a_0 z^s + \cdots + a_s = 0, \quad a_i \in K[x_1, \dots, x_n],$$

that determines an algebraic integral, over the field $R(\mathfrak{p}/K)$ breaks into several equations such that the coefficients of at least one of them are rational integrals of the system.

Remark 4.1. Such assertions are used when searching for algebraic integrals of dynamical systems, and hence one usually proves them specially for Hamiltonian systems; see, for example, [22].

Proof. When an equation

$$a_0 z^s + \cdots + a_s = 0, \quad a_i \in K[x_1, \ldots, x_n],$$

breaks in the field $R(\mathfrak{p}/K)$ into several equations, then one of them, say

$$z^m + b_1 z^{m-1} + \cdots + b_m = 0, \quad b_i \in R(\mathfrak{p}/K),$$

has a fixed root $z = c$. Then for every solution $q \in S(\mathfrak{p}/K)$ on which the denominators b_i do not vanish,

$$c^m + b_1(\dot{q})c^{m-1} + \cdots + b_m(\dot{q}) = 0;$$

differentiating this equality with respect to t, we get

$$\frac{db_1(\dot{q})}{dt} c^{m-1} + \cdots + \frac{db_1(\dot{q})}{dt} = 0.$$

Since the system under consideration is closed, this relation can be written as

$$Db_1(\dot{q})c^{m-1} + \cdots + Db_m(\dot{q}) = 0,$$

which means that the resultant r of the system

$$\{z^m + b_1 z^{m-1} + \cdots + b_m = 0, \quad Db_1 z^{m-1} + \cdots + Db_m = 0\}$$

vanishes at every point $\dot{q}$. By the assumption of the theorem, the system of differential equations is totally consistent over K, hence it follows that the resultant vanishes at all points of $V(\mathfrak{p}/K)$. But then an irreducible equation of order m has common roots with an equation of smaller order, which can happen only when all the coefficients of this equation vanish, i.e.,

$$Db_1 = \cdots = Db_m = 0,$$

or

$$\frac{db_1(\dot{q})}{dt} = \cdots = \frac{db_1(\dot{q})}{dt} = 0,$$

at every point $q \in S(\mathfrak{p}/K)$. □

The proof of the theorem demonstrates that when integrating a system of differential equations it is sufficient to restrict oneself to rational integrals of motion. This theorem can easily be extended to the family of Puiseux extensions from Definition 7.

4.2. The field of rational integrals and its coefficients. If we add the numbers from $\mathbb{C}$ to the rational integrals of the system (1), we get a subfield in the field of rational functions on $V(\mathfrak{p}/K)$, which we call the field of rational integrals and denote by $I(\mathfrak{p}/K)$.

Theorem 5. *Every rational function on the manifold $V(\mathfrak{p}/K)$ can be given by a pair $g : h$ of polynomials from $K[x_1, \ldots, x_{2n}]$ whose coefficients belong to the field generated over k by the values of this function in an appropriate finite set of points of the manifold*

$$V(\mathfrak{p}/K) \cap A_k^{2n} = V(\mathfrak{p}/k).$$

Proof. Take a representation of the function r under consideration as the ratio of polynomials $g : h$, from their monomials $x_1^{n_1} \ldots x_{2n}^{n_{2n}}$ choose those that are linearly independent modulo $\mathfrak{p}^K$ over K, and denote them by $m_1, \ldots, m_u$. Any element of the ring $K[x_1, \ldots, x_{2n}]/\mathfrak{p}^K$ has at most one representation as a linear combination of these monomials with coefficients from K. In particular,

$$g = \sum g_i m_i, \quad h = \sum h_i m_i.$$

We will prove the theorem if we express g_i and h_i in terms of the values of r at points of $V(\mathfrak{p}/k)$.

Take an arbitrary point q_1 from $V(\mathfrak{p}/k)$; then

$$\sum (g_i - r(q_1)h_i) \cdot m_i(q) = 0,$$

which gives linear homogeneous equations on the coefficients g_i and h_i. Then take a point q_2 and add several more equations to the system which do not follow from the other ones. We cannot continue this process indefinitely, since otherwise we would obtain a linear homogeneous system with more equations than unknowns, so that $h = 0$, which is impossible. Thus at step j we obtain a system of homogeneous linear equations such that if the coefficients satisfy this system, then there is no point q_{j+1} that would violate the relation

$$\sum (g_i - r(q_{j+1})h_i) \cdot m_i(q_{j+1}) = 0.$$

In other words, if the coefficients g_i, h_i satisfy the system of linear homogeneous equations constructed in this way, then the function g/h coincides with r at all points of $V(\mathfrak{p}/k)$.

If these functions do not coincide at some point q from $V(\mathfrak{p}/K)$, then there is a value $t = b$ such that

$$\frac{g(q)}{h(q)} - r(q)\bigg|_{t=b} \neq 0.$$

But then these functions cannot coincide at a point p from $V(\mathfrak{p}/k)$ at which

$$p|_{t=b} = q|_{t=b},$$

which is impossible. Thus g/h coincides with r at all points of $V(\mathfrak{p}/K)$ where they are defined.

It remains to observe that the coefficients of the system of linear homogeneous equations for g_i, h_i are the values of monomials at points of $V(\mathfrak{p}/k)$, i.e., elements of the field k, or products of such monomials with the values of the function at $q_1, \dots, q_j$. Hence we can find a solution of this system in the field generated over k by the elements $r(q_1), \dots, r(q_j)$. □

Definition 10. *We say that the coefficients of elements of a subfield P of the field $R(\mathfrak{p}/K)$ lie in a field k' if this field lies between k and K and any function r from P at some open set in $V(\mathfrak{p}/k)$ takes values in k'. The smallest field that contains the coefficients of P is called the* field of coefficients *of the field P and denoted by* $\mathrm{coef}(P)$.

Definition 11. *The field of coefficients of integrals of a system of differential equations is called the* field of transcendentals introduced by integrating this system. *The elements of a transcendence basis of this field over the field of basic functions are called the transcendentals introduced by integrating, and the transcendence degree is called the number of transcendentals introduced by integrating.*

This definition allows us to give a rigorous formulation of Problem 2.

Problem 3. *Enumerate all transcendentals introduced by integrating systems of differential equations.*

4.3. Properties of the field of integrals. If rational integrals of a totally consistent closed system of differential equations are algebraically dependent over a field K, then they are dependent also over the field of constants of the field K. Indeed, if integrals $r_1, \dots, r_s$ of the system satisfy an irreducible (over K) relation

$$\sum a_{n_1,\dots,n_s} r_1^{n_1} \dots r_s^{n_s} = 0, \quad a_{n_1,\dots,n_s} \in K,$$

then

$$\sum \dot{a}_{n_1,\dots,n_s} r_1^{n_1} \dots r_s^{n_s} = 0,$$

so that the coefficients $a_{n_1,\dots,n_s}$ necessarily belong to the field of constants.

This observation allows one to characterize the field of integrals as a classical object of algebraic geometry.

Theorem 6. *If a totally consistent closed system of differential equations allows m rational integrals, then its field of integrals is isomorphic to the field of rational functions on a hypersurface in the affine space of dimension m over the field $\mathbb{C}$.*

Proof. The transcendence degree of the field $R(\mathfrak{p}/K)$ over K is finite, it coincides with the order of the system (1). Thus there are finitely many integrals algebraically independent over K, say $r_1, \dots, r_s$, and any other integral r is connected with them by an algebraic relation whose coefficients lie in K, and hence also in $\mathbb{C}$. Therefore, the field of integrals $I(\mathfrak{p}/K)$ is an algebraic extension of the field $\mathbb{C}(r_1, \dots, r_s)$.

Since the field $R(\mathfrak{p}/K)$ is finitely generated over K, the composite of K and $I(\mathfrak{p}/K)$ lying between K and $R(\mathfrak{p}/K)$ is also finitely generated over K. If integrals $r', r'', \dots$ are linearly independent over $\mathbb{C}(r_1, \dots, r_s)$, then they are also linearly independent over the composite of $\mathbb{C}(r_1, \dots, r_s)$ and K. Since the composite of K and $I(\mathfrak{p}/K)$ is finitely generated over K, there cannot be infinitely many such elements, hence the field $I(\mathfrak{p}/K)$ is finite over $\mathbb{C}(r_1, \dots, r_s)$, and, by the primitive element theorem, in $I(\mathfrak{p}/K)$ there is an element r_{s+1} such that adding it to the field $\mathbb{C}(r_1, \dots, r_s)$ gives the whole field $I(\mathfrak{p}/K)$. □

Remark 4.2. It is substantial for the proof that we work with fields rather than rings; otherwise we would face the 14th Hilbert problem.

As a consequence of Theorems 5 and 6 we have the following.

Theorem 7. *The field of transcendentals introduced by integrating a totally consistent and closed system of differential equations is finitely generated over the field k. A transcendence basis of this field is given by the values of appropriate integrals $r_1, \dots$ at appropriate points $q', \dots$ of the manifold $V(\mathfrak{p}/k)$.*

From the equation $Dr = 0$ it follows that at any point q in $V(\mathfrak{p}/k)$

$$\frac{dr(q)}{dt} = \frac{\partial r}{\partial t}(q) + \sum_i \frac{\partial r}{\partial x_i}(q) \cdot \dot{q}_i = -\sum \frac{\partial r}{\partial m_j} Dm_j + \sum_i \frac{\partial r}{\partial x_i}(q) \cdot \dot{q}_i.$$

By Definition 5, for any monomial m we have $Dm_j \in R(\mathfrak{p}/k)$, hence the function $r(q)$ and its derivative belong to the field of coefficients of integrals of the system. Thus Theorem 7 implies the following theorem.

Theorem 8. *The field of transcendentals introduced by integrating a totally consistent and closed system of differential equations is closed under the differentiation with respect to t.*

4.4. Automorphisms of the field of transcendentals introduced by integrating. Let us now turn to automorphisms of the field of transcendentals; for this we use constructions usual for all Liouville theories which go back to Liouville's paper on integration in elementary functions [28], see also [29–31].

Theorem 9. *The group of differential k-automorphisms of the field of transcendentals introduced by integrating a totally consistent system of differential equations* (1) *closed over K is contained in the group of $\mathbb{C}$-automorphisms of the field of integrals.*

Proof. (i) Every differential k-automorphism of a field k' lying between k and K can be extended to an automorphism of the field of rational functions on $V(\mathfrak{p}/K)$ with coefficients in k'.

Let T be a differential k-automorphism of the field k', i.e., an automorphism commuting with the differentiation of the field k',

$$T\frac{d\alpha}{dt} = \frac{dT\alpha}{dt},$$

and leaving the elements of the field k unchanged. This automorphism can be extended to $k'[x_1, \ldots, x_{2n}]$. The generators of the ideal $\mathfrak{p}$ have coefficients in k, and hence remain fixed, thus $T\mathfrak{p} = \mathfrak{p}$.

The automorphism T can be extended to an automorphism of the field of rational functions on $V(\mathfrak{p}/K)$ with coefficients in k' as follows. By Theorem 5, a rational function r can be written as the ratio $g : h$ of polynomials from $k'[x_1, \ldots, x_{2n}]$. Since $h \notin \mathfrak{p}$ implies $Th \notin T\mathfrak{p} = \mathfrak{p}$, the ratio $Tg : Th$ also determines a rational function on $V(\mathfrak{p}/K)$ with coefficients in the field k'. Before we call it the image Tr, we must prove that this function does not change if we use another representation of r as the ratio of polynomials. Since T leaves the elements of k unchanged, at a point q of the manifold $V(\mathfrak{p}/k)$ the function given by the ratio $Tg : Th$ takes the value $T(r(q))$, and by Theorem 5 these values uniquely determine a rational function.

(ii) The extension of the automorphism T constructed above commutes with the differentiation D of the field $R(\mathfrak{p}/K)$.

We use the same construction as in the proof of Theorem 8: if a rational function is represented as a ratio

$$r = \frac{g_1 m_1 + \ldots}{h_1 m_1 + \ldots},$$

where $m_1, \ldots$ are monomials linearly independent over K modulo $\mathfrak{p}$, then

$$Dr = \frac{\partial r}{\partial m_1} Dm_1 + \cdots + \frac{\partial r}{\partial t}.$$

Since T leaves the elements of the field k unchanged, we have

$$TDm_i = DTm_i = Dm_i.$$

Since T commutes with the differentiation with respect to t,

$$T\frac{\partial r}{\partial t} = \frac{\partial Tr}{\partial t}.$$

Thus the extension of T to the rational functions with coefficients in k' commutes with D.

(iii) If T_1 and T_2 are two automorphisms of the field k', then, as we can see from the construction in (i), the extension of their product T_1T_2 coincides with the product of their extensions.

(iv) If k' is the field of transcendentals, then the extension described in (i) determines a homomorphism φ of the group of differential k-automorphisms of the field k' to the group of $\mathbb{C}$-automorphisms of the field of integrals.

Every differential automorphism T of the field of transcendentals k' can be extended to an automorphism of the field of rational functions with coefficients in k'. All integrals of the system lie in this field, and the automorphism sends integrals to integrals, since $Dr = 0$ implies $DTr = TDr = 0$. Thus T can be extended to an automorphism of the field $I(\mathfrak{p}/K)$. By assumption, $\mathbb{C}$ is the field of constants of k, hence T leaves the constants unchanged and permutes the integrals of the system.

(v) The homomorphism constructed in (iv) is a monomorphism.

An automorphism T of the field of transcendentals different from the identity one moves at least one element of this field, and hence at least one of the elements

$$\alpha_1 = r_1(q_1), \ldots$$

that generate it over the field k changes under the action of T. This means that the value of at least one integral at least at one point of the manifold $V(\mathfrak{p}/k)$ changes under the action of T. □

The field of transcendentals k' introduced by integrating a system is finitely generated over the field of basic functions k. Denote a transcendence basis of this field by $\alpha_1, \ldots, \alpha_u$; the whole field k' can be obtained from $k(\alpha_1, \ldots, \alpha_u)$ by adding one primitive element α_{u+1} connected with $\alpha_1, \ldots, \alpha_u$ by an irreducible (over k) relation

$$g(\alpha_1, \ldots, \alpha_{u+1}) = 0, \quad g \in k[y_1, \ldots, y_{u+1}].$$

Denote the prime ideal generated by the left-hand side of this equation by $\mathfrak{q}$. Since the field k' is closed under the differentiation with respect to t, we can find elements f_i of the ring $k(y_1, \ldots, y_u)[y_{u+1}]$ such that

$$\dot{\alpha}_i = f_i(\alpha_1, \ldots, \alpha_{u+1}).$$

The differential k-automorphism T sends these relations to

$$\frac{dT\alpha_i}{dt} = f_i(T\alpha_1, \ldots, T\alpha_{u+1}).$$

In other words, differential k-automorphisms send solutions of the system of differential equations

$$\dot{y}_i = f_i(y_1, \ldots, y_{u+1}) \tag{3}$$

on the manifold $V(\mathfrak{q}/K)$ to solutions. Among the coordinates of a point on $V(\mathfrak{q}/K)$, at most u can be algebraically independent over k. The solution under consideration

$$y_1 = \alpha_1, \ldots, y_{u+1} = \alpha_{u+1}$$

of the system (3) has a remarkable property: its coordinates have the largest possible transcendence degree over k. Such a solution will be called *a totally transcendental solution* of the system.

Definition 12. *An extension k' of the field k of basic functions is called normal in the field K if*

(1) *this field is generated over k by a totally transcendental solution of some system of differential equations over k,*
(2) *all totally transcendental solutions of this equation in the field K belong to the field k'.*

This definition is given by analogy with that of a normal extension in Galois theory; no confusion is possible, since k' is not an algebraic extension of k.

Theorem 10. *The field of transcendentals introduced by integrating a totally consistent and closed system of differential equations is a normal extension of the field of basic functions.*

Proof. Consider two totally transcendental solutions

$$y_1 = \alpha_1, \ldots, y_{u+1} = \alpha_{u+1} \quad \text{and} \quad y_1 = \beta_1, \ldots, y_{u+1} = \beta_{u+1}$$

of the system (3), the first one generating a field $k' = \operatorname{coef}(I(\mathfrak{p}/k))$ and the second one generating a field k''. Denote by T the homomorphism $k' \to k''$ that sends α_i to β_i. The commutation relations established in the proof of Theorem 9 remain valid, hence the extension of T sends integrals to integrals. The field of coefficients of these integrals at any case belongs to the field of coefficients of all integrals, i.e., k' and k'' satisfy $k'' \subset k'$ as subfields in K.

On the other hand, $\beta_1, \ldots, \beta_{u+1}$ are certainly connected by the relation

$$g(\beta_1, \ldots, \beta_{u+1}) = 0$$

over k, hence $\beta_1, \dots, \beta_u$ cannot be connected by another additional relation and, therefore, form a transcendence basis of k''. Since g is irreducible over k,

$$k' = k(\alpha_1, \dots, \alpha_u)[\alpha_{u+1}], \quad k'' = k(\beta_1, \dots, \beta_u)[\beta_{u+1}],$$

and the homomorphism T is a natural homomorphism of these fields. □

Theorem 11. *If integrating a totally consistent system of differential equations closed over a Puiseux extension introduces u transcendentals, then the field of its integrals has a u-parameter group of* $\mathbb{C}$*-automorphisms, which is an extension of the group of differential automorphisms of the field of coefficients of the field of integrals of the system.*

Proof. Since the field of transcendentals is a normal extension of the field of basic functions, every totally transcendental solution of the system (3) in a Puiseux field generates a differential k-automorphism of this field. By Theorem 9, every such automorphism of the field of transcendentals generates an automorphism of the field of integrals.

Since the point $(\alpha_1, \dots, \alpha_{u+1})$ belongs to the domain of definition of the right-hand sides of the system (3), there is a value $t = b$ such that

$$f_i(\alpha_1, \dots, \alpha_{u+1})|_{t=b} \in \mathbb{C}$$

for all i. But then the Cauchy problem for the system (3) with the initial conditions

$$y_1 - \alpha_1|_{t=b} = c_1, \dots, y_u - \alpha_u|_{t=b} = c_u$$

has a solution in the Puiseux field that depends on u parameters ranging in a neighborhood of zero. For $c_1 = \cdots = c_u = 0$, this solution coincides with $\alpha_1, \dots, \alpha_{u+1}$, and hence its coordinates cannot be connected by any other relation apart from $g = 0$ and its consequences. Thus the Cauchy theorem does not give a totally transcendental solution only for those values of the parameters $c_1, \dots, c_u$ that do not have a condensation point at zero. Removing them, we obtain a u-parameter family of totally transcendental solutions of the problem (3). □

As in Galois theory one reduces the analysis of the solvability of algebraic equations of order n to the analysis of subgroups of the finite group of permutations of n elements, the proven theorem reduces the study of the solvability of a system of differential equations to the study of infinite groups of automorphisms of algebraic manifolds.

5. Conclusion

Combining Theorems 6 and 11, we see that *the field of integrals of a system of differential equations is equivalent to the field of rational functions on a hypersurface having a continuous group of birational* $\mathbb{C}$*-automorphisms whose dimension coincides with the number of transcendentals introduced by integrating the system.*

It is well known that the group of automorphisms of an arbitrary algebraic curve of degree n is finite (see, e.g., [32]); thus in the suggested version of Galois theory, the analog of a solvable group among finite groups is a continuous group of automorphisms among automorphisms of algebraic manifolds. The description of manifolds over $\mathbb{C}$ that have infinite groups of birational automorphisms was the subject of many investigations by Italian geometers [33, no. 39], [34,35], and the proved theorem opens the way for applying them to the theory of differential equations.

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