On M.N. Lagutinski method for integration of ordinary differential equations

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20 apr. 2016, ver. April 20, 2016

I. Beaune's problem

Florimond de Beaune and Descart, 1640

In modern terms Florimond de Beaune asked Descartes how to find an algebraic integral of the differential equation

$$y' = y.$$

And Descartes wanted to find the solution by selection of coefficients in equation

$$y^m = a_0 + a_1 x + \cdots + a_n x^n,$$

but stopped on $n, m \simeq 100$.

Main Problem

Problem (de Beaune, 1640)

For given ordinary differential equation

$$pdx + qdy = 0, \quad p, q \in \mathbb{Q}[x, y],$$

the problem, attributed below to Beaune, consists of finding an integral w from $\mathbb{C}(x, y)$.

Test for Maple

Let u, v be polynomials from $\mathbb{Z}[x, y]$. Then quotient $w = u/v$ is an integral of eq.

$$\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) dx + \left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) dy = 0 \qquad (1)$$

Put for example

$$u = x^4y^{11} + x^{10} + y, \quad v = 2x^{11}y^8 + y^{10} + 3xy.$$

Standard solver `dsolve` in Maple can solve the eq. (1) in quadratures

$$\int rdx + sdy = C, \quad r, s \in \mathbb{Q}(x, y),$$

but the explicit formula occupies two screens and Maple can't calculate this integrals.

Triviality of the test

Generated ODE

$$p(x,y)dx + q(x,y)dy = 0, \quad p,q \in \mathbb{Q}[x,y]$$

has several linear independent integrating divisors in the ring $\mathbb{Q}[x,y]$.
Integrating divisor χ is any nonconstant solution of eq.

$$\chi^2 \frac{\partial}{\partial x}\frac{q}{\chi} = \chi^2 \frac{\partial}{\partial y}\frac{p}{\chi}.$$

So we can find an integrating divisor in the ring $\mathbb{Q}[x,y]$ by indeterminate coefficients method from a system of linear algebraic equation.

Maple fail

Maple looks for an integrating divisor in the ring $\mathbb{C}[x, y]$ as a part of 2nd DETools algorithm [Cheb-Terrab, 1996].
However:

- `symgen` returns two integrating divisors, so its ratio is required integral,
- `dsolve` ignores 2nd divisor and writes a quadrature

$$\int \frac{p(x,y)dx + q(x,y)dy}{\chi_1(x,y)} = C,$$

- `int` can't calculate this integrals in finite terms.

So in general `symgen` can solve the problem, but `dsolve` damages the result.

Maple fail in case of multiple factors

If p and q have common factors there are insuperable difficulties.
In the beginning Maple reduces common factors in ODE

$$p(x,y)dx + q(x,y)dy = 0, \quad p,q \in \mathbb{C}[x,y].$$

Reducible equation hasn't any integration divisors in the ring $\mathbb{C}[x,y]$ and finding of integration divisor in $\mathbb{C}(x,y)$ lids to nonlinear algebraic equation for coefficients.
For ex.

$$u = (x^2+y)^5(x-y^6+1)+1, \quad v = (13xy^8+y^5+3xy+2)(x^2+y)^4,$$

`symgen` finds only one divisor and can't solve the problem.

Main difficulties

If eq.

$$p(x,y)\cdot dx + q(x,y)\cdot dy = 0, \quad p,q \in \mathbb{Q}[x,y],$$

has a rational integral, then integral curves consist a irreducible linear pencil

$$u(x,y) + c\cdot v(x,y) = 0, \quad u,v \in \mathbb{C}[x,y],\ c \in \mathbb{C}.$$

Poincaré, 1890 [Œuvres, vol. 3.]: How to estimate the pencil order? — This is an open question and appears in [Chèze, 2010]. In all algorithms for solution of Beaune's problem the user must give a bound N for degree of required integral.

Two methods of solution

- Method of Jacques-Arthur Weil, 1985, use a solution of Cauchy problem

$$\begin{cases} \frac{dy}{dx} = -\frac{p}{q} \\ y(0) = c \end{cases} \mod x^{N^2+1}$$

in $\mathbb{Q}(c)[x]$ and have several realizations [Bostan et all, 2013].

- Method of M.N. Lagutinski, 1913, is a generalization of Kramer's formulas from linear algebra.

II. Lagutinski theory and modern computer algebra

Mikhail Nikolaevich Lagutinski (1871 – 1915)

On the eve of World War I M.N. Lagutinski worked at Kharkov university and died during academical training in Germany.

- Bio in Russian: Istoriko-matematicheskie issledovanija. 2001. T. 6 (41). P. 111-127.
- Bio in French: Historia Mathematica. Volume 25, Issue 3, August 1998, Pag. 245-264.
- Works: `http://www.mathnet.ru/php/person.phtml?option_lang=eng&personid=39928`

Lagutinski method was partially reopened in 1990-2000; his works became known thanks to activity of Jean-Marie Strelcyn (Université de Rouen) [J.-M. Strelcyn, 2008], [Chèze, 2010].

Differential ring with basis

Let R be a ring with differentiation D and constant field k.
Countable set

$$B = \{\varphi_1, \varphi_2, \dots,\}, \quad \varphi_n \in R,$$

is called a basis of ring R if:

1. any element ψ from the ring R can be describe as linear combination of elements from B under constant field:

$$\psi = c_{n_1}\varphi_{n_1} + \cdots + c_{n_m}\varphi_{n_m}, \quad c_{n_1}, \dots, c_{n_m} \in k.$$

2. for any integers i, j there is an integer $n > \max(i, j)$ such that

$$\varphi_i \varphi_j = \varphi_n.$$

Maximum from numbers $n_1, \dots, n_m$ we will called as order of ψ with respect to the basis B.

Example in Sagemath

Consider the ring

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
```

with the differentiation

```
sage: D=lambda phi: y*(x+1)*diff(phi,x)+(y
    ^2+x+2)*diff(phi,y)
```

Put the standard glex-basis as

```
sage: B = sorted(((1+x+y)^5).monomials(),
    reverse=0)
```

So we have

```
sage: B[3]
y^2
sage: D(x^2+y)
2*x^2*y + 2*x*y + y^2 + x + 2
```

Lagutinski determinant

Definition

Determinant of the matrix

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \\ D\varphi_1 & D\varphi_2 & \dots & D\varphi_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}\varphi_1 & D^{n-1}\varphi_2 & \dots & D^{n-1}\varphi_n \end{pmatrix} \tag{2}$$

is called Lagutinski determinant of degree n and will denoted bellow as Δ_n.

So in our example

```
sage: lagutinski_det(3,B)
x^3 + x*y^2 + 5*x^2 + y^2 + 8*x + 4
```

Partial integral

Element u from the ring R is called a partial integral, if there is an elements v from R such that

$$Du = v \cdot u.$$

If R is a polynomials ring, then partial integrals are well-known as Darboux polynomials.

Theorem

If there exist an partial integral u of degree n, then Lagutinski determinant of degree n is equal to zero or has u as a factor.

Partial integral

In our example

```
sage: D(x+1).factor()
y * (x + 1)
```

so $x+1$ is Darboux polynomial of degree 3. By the theorem under $n > 2$ all Lagutinski determinants have factor $x+1$:

```
sage: lagutinski_det(3,B).factor()
(x + 1) * (x^2 + y^2 + 4*x + 4)
sage: lagutinski_det(4,B).factor()
(6) * y * (x + 1) * (y^2 + x + 2) * (y^2 +
    2*x + 3) * (x^2 + y^2 + 4*x + 4)
```

The second repeating factor is also Darboux polynomial:

```
sage: D(x ^2 + y ^2 + 4* x + 4).factor()
(2) * y * (x^2 + y^2 + 4*x + 4)
```

Rational integral

Rational integral of differentiation D is called a pair of elements u, v from the ring R such that

$$uDv - vDu = 0.$$

If the ring is integral, that we will write this pair as u/v.

Theorem

If one from Lagutinski determinants is equal to zero, then there exist rational integral.

Example

For our example:

```
sage: lagutinski_det(5,B)==0
False
sage: lagutinski_det(6,B)==0
True
sage: lagutinski_integral(6,B)
(-54*x^2 + 18*y^2 - 72*x)/(-18*y^2 - 36*x -
    54)
```

Check:

```
sage: D(lagutinski_integral(6,B))
0
```

The first argument in `lagutinski_integral` has to be equal a minimal degree of zero Lagutinski determinant.

Converse

If there exits a rational integral u/v, then there are several partial integrals of the same degree, namely

$$u + cv \quad \forall c \in k.$$

Thus all Lagutinski determinant of big degree has several factors.

Theorem

Let k be an infinite field and R be a ring with unique factorization of elements. If there exist rational integral, then all Lagutinski determinant of sufficiently big degree are equal to zero.

The strong assumption about k and R appears only in this theorem.

A necessary and sufficient condition for existence

Let R be a polynomial ring over $\mathbb{Q}$.

Theorem

1. *A rational integral exist iff all Lagutinski determinants of sufficiently big degree are equal to zero:*

$$\exists N \in \mathbb{N} : \quad \forall n \geq N : \quad \Delta_n = 0.$$

2. *A rational integral of degree N exist iff $\Delta_N = 0$.*

Furthermore, if we have $\Delta_N = 0$, then we can calculate rational integral. But if we have

$$\Delta_1 \neq 0, \, \Delta_2 \neq 0, \ldots, \Delta_{100} \neq 0,$$

then we can't tell that the rational integral of any degree doesn't exist.

Problem about trinomials

Linear combination of three monomials is called a trinomial.

Problem

For given ODE the problem consists of finding an integral which is ratio of two trinomials of degree N or smaller.

To solve this problem we need calculate a big set of Lagutinski determinants of third degree. For this purpose we can use parallel programming model in Sage.

Example

Lagutinski_micronomial function returns the number of required trinomial in the given set B_3 of a trinomials:

```
sage: R.<x,y> = PolynomialRing(QQ, 2)
sage: B= sorted(((1+x+y)^5).monomials(),
   reverse=0)
sage: B3=[[n,m,k] for n in B for m in B for
    k in B if (n<m<k)]
sage: D= lambda u: (3*x^4 + 2*y)*x*y^2*diff
   (u,x)+ (5*x^4 + y)*y^3*diff(u,y)
sage: lagutinski_micronomial(B3)
678
sage: lagutinski_integral(3,B3[678])
(7*x^5 + 7*x*y)/(7*y^3)
sage: D(7*y^3/(-7*x^5 - 7*x*y))
0
```

III. A necessary condition for existence of rational integral

Contractor

Let

$$B = \{\varphi_1, \varphi_2, \dots\}$$

be a basis of the ring R. If

$$\psi = c_{n_1}\varphi_{n_1} + \cdots + c_{n_m}\varphi_{n_m}, \quad (n_1 < n_2 < \cdots < n_m)$$

we say that $c_{n_1}\varphi_{n_1}$ is a main member in ψ and write

$$\psi = c_{n_1}\varphi_{n_1} + o(\varphi_{n_1}).$$

The differentiation D is said to be a contractor with respect to the basis B if

$$D\varphi_n = c_n\varphi_n + o(\varphi_n), \quad c_n \in k.$$

Constants $c_1, c_2, \dots$ are called contraction indexes.

A necessary condition for existence

Lemma

If D is contractor with respect to the basis

$$B = \{\varphi_1, \varphi_2, \dots\},$$

then

$$\Delta_n = W(c_1, c_2, \dots, c_n) \prod_{i=1}^{n} \varphi_i + o\left(\prod_{i=1}^{n} \varphi_i\right),$$

where W is Vandermonde determinant.

Theorem

Let differentiation D be a contractor. If there exists rational integral, then there aren't two equal contraction indexes:

$$c_i \neq c_j \quad \forall i \neq j.$$

Brio et Bouquet equation

Let the ring be $\mathbb{C}[x,y]$. The ODE

$$(ay + cx + \dots)dx + (bx + \dots)dy = 0, \quad a, b, c \in \mathbb{C} \tag{3}$$

is well-known as Brio et Bouquet equation [Ince, n. 12.6]. Integrals of this eq. are also integrals of the contractor

$$D = (ay + cx + \dots)\frac{\partial}{\partial y} - (bx + \dots)\frac{\partial}{\partial x}.$$

Theorem

If Brio et Bouquet equation (3) has rational integral in field $\mathbb{C}(x,y)$, then complex numbers a and b are linear dependent over field $\mathbb{Q}$.

Example

General solution of LDE

$$(ay + cx)dx + bxdy = 0$$

is

$$x^{a/b}\left(y + \frac{cy}{b+a}\right) = C.$$

This integral is algebraic iff $a/b \in \mathbb{Q}$.
Previous theorem generalizes this statements on all Brio et Bouquet equation

$$(ay + cx + \dots)dx + (bx + \dots)dy = 0.$$

Without any power series decompositions!

General case

If ODE

$$p(x, y)dx + q(x, y)dy = 0$$

has singularity at point $(x, y) = (x_0, y_0)$, then we can lead this eq. to Brio et Bouquet equation by linear transformation. Now if we recall previous theorem we get simple necessary condition for existence of a rational integral.
This condition can check by help my `lagutinski_ab` function.

Example 1.

For LDE

$$(x+y)dx + xdy = 0$$

we have:

```
sage: x,y=var('x,y')
sage: lagutinski_ab(x+y,x)
True
```

So rational integral may exist. There is

$$y = -\frac{x}{2} + \frac{C}{x}.$$

Example 2.

For ODE

$$(2 - x^2 - y^2)dx + (x - y)dy = 0$$

we have:

```
sage: lagutinski_ab(2-x^2-y^2,x-y)
False
```

So there is no rational integral in the field $\mathbb{C}(x, y)$.

Maple 17 (X86 64 LINUX)

```
> dsolve((2-x^2-y(x)^2) + (x-y(x))*diff(y(x),x)=0,y(x));
memory used=224.7MB, alloc=88.8MB, time=3.15
memory used=256.5MB, alloc=88.8MB, time=3.45
memory used=298.9MB, alloc=88.8MB, time=3.82
memory used=376.1MB, alloc=105.7MB, time=4.60
```

The end.

Calculations made in `SageMath Version 7.1, Release Date: 2016-03-20`.
See additional materials on `http://malykhmd.neocities.org`.