Weierstrass Theory of Abelian Integrals and its Realization in Sage

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Abelian integrals

It is well-known, that abelian integral is an integral of the form

$$\int R(x, y)dx,$$

here R is an arbitrary rational function of the two variables x and y related by the equation

$$f(x, y) = 0,$$

where f is an irreducible over $\mathbb{C}$ polynomial from $\mathbb{Q}[x, y]$.

Theory of abelian integrals

Mathematicians of 19th century considered the theory of abelian integrals as the necessary completion of mathematical analysis, but after WWI works in this theory have died away. So Felix Klein wrote in 1926:

> *Als ich studierte, galten die Abelschen Funktionen (...) als der unbestrittene Gipfel der Mathematik, und jeder von uns hatte den selbstverstandlichen Ehrgeiz hier selbst weiterzukommen. Und jetzt? Die junge Generation kennt die Abelschen Funktionen kaum mehr.*

Abelian integrals in CAS

In modern CAS there are few packages for work with abelian integrals:

- `Algcurves` for Maple (M. van Hoeij et all.),
- `Casa` for Maple V (Franz Winkler et all.),
- `Curve generic` for Sage.

Standard sources about the theory of abelian integrals for modern authors are:

- Backer, 1897; last ed. 2008 (!),
- Tikhomandritsky, 1895

Algcurves for Maple

The package Algcurves can calculate the genus of a given curve or a basis for the linear space of differentials of the first kind.

```
> with(algcurves):
> f:=y^2-x^3-2*x-3;
          3    2
  f := -x  + y  - 2 x - 3
> genus(f,x,y);
  1
> differentials(f,x,y);
   dx
 [----]
   y
```

In other words, the genus of the given elliptic curve is equal to 1, and the basis for the called linear space consists of one element $\int dx/y$.

Weierstrass lectures

Weierstrass didn't publish his results after 1870[1], so authors of well-known reviews on the theory of abelian integral used incomplete students manuscripts.

Main idea.

For realization of Weierstrass ideas in CAS we have to work over $\overline{\mathbb{Q}}$ in contrary of "kroneckerian" algorithms over $\mathbb{Q}$.

In 2007 Carl Witty has added to Sage the user friendly realization of the field of Algebraic Numbers (`QQbar`). So we wont to illustrate Weierstrass theory by calculations in Sage.

[1] See biographical details in [Klein, 1926; G.I. Sinkevich, 2015].

Hauptfunktion

Definition

If (x', y') is a point on this curve then there is such function $H \in \mathbb{C}(x, y)$ that (x', y') is a simple pole of H and the residue at this point is equal 1. Such function with minimal order $r = 1 + p$ is called a fundamental function (Hauptfunktion) and the number p is called a genus (Rang) of curve.

Trivial statement of the existence of the fundamental function is the unique existence theorem in Weierstass lectures.

Hauptfunktion for elliptic curve

In many cases we can write expression for fundamental function explicitly, so for elliptic curve

$$y^2 = a_0 y^3 + a_1 y^2 + a_2 y + a_3$$

the fundamental function is equal

$$\frac{1}{2y'} \frac{y + y'}{x - x'}.$$

This function has pole at (x', y') and at infinity, the genus is equal to 1.

Local uniformization

The neighborhood of any point (a, b) on an algebraic curve $f(x, y) = 0$ can be described as arch or union of several arches and each of these arches can be presented parametrically

$$x = a + a't + \dots, \quad y = b + b't + \dots.$$

In Weierstrass lectures this series as x_t, y_t and point (a, b) as the center of called arch or arches.
The algorithm for calculation of this series from 1st chapter of Weierstrass lectures uses on each step solutions of univariable algebraic equations, so we can write literal its realization in Sage over `QQbar`.

Example

At infinity we have simple pol:

```
sage: var('x,y,t,xt,yt,xx,yy')
(x, y, t, xt, yt, xx, yy)
sage: load('sage/weierstrass.sage')
None
sage: f= y^2-x*(x-1)*(x-2)
sage: H= (1/2/yy)*(y+yy)/(xx-x)
sage: L=local_parametrization(f,oo,oo,t,4)
sage: H.subs([x==xt, y==yt]).subs(L[0]).
    series(t,2)
(-1/2/yy)*t^(-1) + (-1/2/yy) + (-1/2*xx/yy)
    *t + Order(t^2)
```

Duality

By analogy with Green's function, Weierstrass considers not $H(x, y)$, but

$$H(x, y; x', y') \cdot dx'.$$

This dual construction is a rational function with respect to (x, y) and a differential with respect to (x', y').
If we known Haupfunktion for the given curve we have explicit expressions for the abelian integrals of 1-3 kinds.

Integrals of 3rd kind

Theorem

Expression of Hauptfunktion in Laurent series with respect to second argument has the form

$$H(x, y; x_t, y_t)\frac{dx_t}{dt} = \frac{\delta}{t} + c_0 + c_1 t + \ldots;$$

where the residue $\delta \neq 0$ iff center of the arch coincides with (x, y) or yet one singular point.

So expression $H(x, y; x', y')dx'$ with respect to second argument is known in other theories as an integral of the 3rd kind (Art).

Integrals of 3d kind for elliptic curve

Expression

$$\frac{y+y'}{x-x'}\frac{dx'}{2y'}$$

with respect to (x', y') has poles at (x, y) and at infinity.

```
sage: (H.subs([xx==xt, yy==yt]).subs(L[0])*
    diff(xt.subs(L[0]),t)).series(t,2)
(-1)*t^(-1) + 1 + (-x + 2)*t + Order(t^2)
```

So residue at infinity is equal -1.

Integrals of 1st and 2nd kinds

Theorem

Let $(a_1, b_1), \dots, (a_p, b_p)$ be poles of the fundamental function with respect to first argument. Coefficients of Laurent series

$$H(x_t, y_t; x', y')dx' = H_n(x', y')dx' \cdot \frac{1}{t} + c_0(x', y') - H_n'(x', y')dx' \cdot t + \dots$$

for arch with center at (a_n, b_n) give us well-known abelian integrals of the 1st and the 2nd kinds.

This means that

$$H_n(x_t, y_t)\frac{dx_t}{dt} = c_0 + c_1 t + \dots, \quad H_n'(x_t, y_t)\frac{dx_t}{dt} = \frac{\delta}{t^2} + c_0 + c_1 t + \dots$$

on any arch and $\delta \neq 0$ iff center of the arch coincides with (a_n, b_n).

Integrals of 1st and 2nd kinds for elliptic curve

For the chosen fundamental function

$$\frac{y+y'}{x-x'}\frac{dx'}{2y'}$$

the set of poles consists from one point (∞,∞).

```
sage: H.subs([x==xt, y==yt]).subs(L[0]).
    series(t,2)
(-1/2/yy)*t^(-1) + (-1/2/yy) + (-1/2*xx/yy)
    *t + Order(t^2)
```

Therefore

$$H_1(x,y)dx = -\frac{dx}{2y}, \quad H_1'(x,y)dx = \frac{xdx}{2y}$$

Integrals of 1st kind for elliptic curve

The differential

$$H_1(x,y)dx = -\frac{dx}{2y}$$

has no singularities on curve. Indeed, at infinity we have

```
sage: H1=-1/2/y
sage: (H1.subs([x==xt, y==yt]).subs(L[0])*
    diff(xt.subs(L[0]),t)).series(t,2)
1 + (-2)*t + Order(t^2)
```

At $(x,y)=(0,0)$

```
sage: M=local_parametrization(f,0,0,t,4)
sage: (H1.subs([x==xt, y==yt]).subs(M[0])*
    diff(xt.subs(M[0]),t)).series(t,2)
(-1/2) + Order(t^2)
```

Integrals of 2nd kind for elliptic curve

The differential

$$H_1'(x,y)dx = \frac{xdx}{2y}$$

has a singularity only at infinity. Indeed, at infinity we have the pole of 2nd degree:

```
sage: HH1=x/2/y
sage: (HH1.subs([x==xt, y==yt]).subs(L[0])*
    diff(xt.subs(L[0]),t)).series(t,2)
(-1)*t^(-2) + Order(t^2)
```

At $(x,y)=(0,0)$ there is no singularity:

```
sage: (HH1.subs([x==xt, y==yt]).subs(M[0])*
    diff(xt.subs(M[0]),t)).series(t,2)
Order(t^2)
```

Fundamental relation

Green's function is symmetric and by analogy Weierstrass proves that

$$\frac{d}{dx_1}H(x_1,y_1;x_2,y_2)-\frac{d}{dx_2}H(x_2,y_2;x_1,y_1)=$$
$$=\sum_{n=1}^{p}H_n(x_2,y_2)H_n'(x_1,y_1)-H_n(x_1,y_1)H_n'(x_2,y_2),$$

This fundamental equation plays the same role in Weierstrass theory that reduction formulas in the theory of rational integrals.

Generalization of Ostrogradski method

For any rational function R we can write abelian integral

$$\int R(x,y)dx$$

as sum of algebraic part $R'(x,y)$, log-part

$$\sum_m c_m H(x_m, y_m; x, y)dx$$

with log-singularities in poles of R and the 3rd part

$$\sum_{n=1}^{p} g_n' \int H_n(x,y)dx - g_n \int H_n'(x,y)dx$$

with simple poles in fixed singularities $(a_1, b_1), \dots, (a_p, b_p)$ of the fundamental function.

What we have?

This 3nd part is an elementary function iff all g_n and g'_n are equal to zero. So we have at once:

- decomposition of given abelian integral into integrals of three kinds,
- conditions for integrating given algebraic function in elementary functions, and
- equivalence between Weierstrass definition of genus and commonly used definition as dimension of linear space of homomorphic abelian integrals.

Calculation of coefficients

In Weierstrass lectures there are explicit formulas for calculation c_m, g_n, g'_n using the coefficients of Laurent series. We will write out these formulas in example bellow.
So Weierstrass wrote explicit formulas where modern authors writes complicated algorithms, see [Devenport, 1985].

Example on reduction

The differential

$$\frac{3x^2 - 6x + 2}{y^3} dx$$

has poles on the elliptic curve

$$y^2 = x(x-1)(x-2)$$

at $y = 0$, so in three points

$$\{(x_\nu, y_\nu)\} = \{(0,0),(0,1),(0,2)\}.$$

The set

$$\{(a_\alpha,\ b_\alpha)\}$$

reduced to one point set $\{(\infty, \infty)\}$.

Coef. c_m

$$c_m = \left[R(x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^{-1}}$$

Residue at the first pole is equal to zero:

```
sage: R=(3*x^2 - 6*x + 2)/y^3
sage: M=local_parametrization(f,0,0,t,4)
sage: (R.subs([x==xt, y==yt]).subs(M[0])*
    diff(xt.subs(M[0]),t)).series(t,4)
2*t^(-2) + (-6)*t^2 + Order(t^4)
```

So $c_1 = 0$ and, analogically, $c_2 = c_3 = 0$.

Auxiliary coefficients

$$-nc_{m,-n} = \left[R(x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^{-n-1}}$$

At the poles we have

```
sage: (R.subs([x==xt, y==yt]).subs(M[0])*
    diff(xt.subs(M[0]),t)).series(t,4)
2*t^(-2) + (-6)*t^2 + Order(t^4)
```

So $c_{m,-n} \neq 0$ iff $n = 1$ and then

$$c_{m,-1} = 2.$$

Coef. g_1

$$g_1 = \sum_{m=1}^{3} \sum_{n>0} c_{m,-n} \left[H_1(x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^{n-1}} =$$
$$= 2 \sum_{m=1}^{3} \left[H_1(x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^0} .$$

At 1st pole we have

```
sage: (H1.subs([x==xt, y==yt]).subs(M[0])*
    diff(xt.subs(M[0]),t)).series(t,4)
(-1/2) + (-3/4)*t^2
```

so

$$\left[H(x_t^1, y_t^1) \frac{dx_t^1}{dt} \right]_{t^0} = -\frac{1}{2}$$

Coef. g_1, no. 2

At 2nd and 3rd poles we have

```
sage: M1=local_parametrization(f,1,0,t,6)
sage: M2=local_parametrization(f,2,0,t,4)
sage: (H1.subs([x==xt, y==yt]).subs(M1[0])*
   diff(xt.subs(M1[0]),t)).series(t,4)
1 + Order(t^4)
sage: (H1.subs([x==xt, y==yt]).subs(M2[0])*
   diff(xt.subs(M2[0]),t)).series(t,4)
(-1/2) + 3/4*t^2
```

so

$$g_1 = 2\left(-\frac{1}{2} + 1 - \frac{1}{2}\right) = 0$$

Coef. g_1'

$$g_1' = \sum_{m=1}^{3} \sum_{n>0} c_{m,-n} \left[H_1'(x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^{n-1}} - R(a_1, b_1) =$$

$$= 2 \sum_{m=1}^{3} \left[H_1'(x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^0} - R(\infty, \infty).$$

For calculation of the value of the function R at infinity we have

```
sage: (R.subs([x==xt, y==yt]).subs(L[0])).
   series(t,4)
Order(t^4)
```

So

$$R(\infty, \infty) = 0.$$

Coef. g_1', no. 2

For calculation of

$$\left[H_1'(x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^0}$$

we have

```
sage: (HH1.subs([x==xt, y==yt]).subs(M[0])*
    diff(xt.subs(M[0]),t)).series(t,4)
1/4*t^2 + Order(t^4)
sage: (HH1.subs([x==xt, y==yt]).subs(M1[0])
    *diff(xt.subs(M1[0]),t)).series(t,4)
(-1) + 1*t^2 + Order(t^4)
sage: (HH1.subs([x==xt, y==yt]).subs(M2[0])
    *diff(xt.subs(M2[0]),t)).series(t,4)
1 + (-5/4)*t^2 + Order(t^4)
```

So $g_1' = 2\,(0 - 1 + 1) = 0.$

The algebraic part

Total:

$$c_1 = c_2 = c_3 = 0, \quad g_1 = g_1' = 0$$

Thus

$$\int \frac{3x^2 - 6x + 2}{y^3} dx$$

is rational function on the curve, namely

$$R' = -\sum_{m=1}^{3} \sum_{n>0} c_{m,-n} \left[H(x, y; x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^{n-1}} =$$

$$= -2 \sum_{m=1}^{3} \left[H(x, y; x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^0} .$$

Example on the reduction: the algebraic part, no. 2

For calculation of

$$\left[H(x, y; x_t^m, y_t^m) \frac{dx_t^m}{dt} \right]_{t^0}$$

we have

```
sage: (H.subs([xx==xt, yy==yt]).subs(M[0])*
    diff(xt.subs(M[0]),t)).series(t,1)
(-1/2*y/x) + Order(t)
sage: (H.subs([xx==xt, yy==yt]).subs(M1[0])
    *diff(xt.subs(M1[0]),t)).series(t,1)
(y/(x - 1)) + Order(t)
sage: (H.subs([xx==xt, yy==yt]).subs(M2[0])
    *diff(xt.subs(M2[0]),t)).series(t,1)
(-1/2*y/(x - 2)) + Order(t)
```

Checking

By help of Weierstrass formulas we have

$$\int \frac{3x^2-6x+2}{y^3}dx = -y\left(-\frac{1}{x}+\frac{2}{x-1}-\frac{1}{x-2}\right) = \frac{2}{y}$$

In this case we can easy check the answer: from

$$y^2 = x(x-1)(x-2)$$

follows that

$$2ydy = (3x^2-6x+2)dx$$

and thus

$$\int \frac{3x^2-6x+2}{y^3}dx = \int \frac{-2ydy}{y^3} = \frac{2}{y}.$$

Conclusion

The reviewed examples show that formulas from Weierstrass lectures can be used for concrete calculations in any realization of the field $\overline{\mathbb{Q}}$.

The main difficulty is search of the fundamental function for given curve. In Weierstrass lectures there is an algorithm for this, but even for general cubic we have complicated expression. So we wont to simplify it in the near future.

Russian retelling of Weierstrass lectures can be found on our site `https://malykhmd.neocities.org/`

The end.

Calculations made in `SageMath version 7.2, Release Date: 2016-05-15`.
See additional materials on `http://malykhmd.neocities.org`.