Finite difference schemes and classical transcendental functions [1]

Ayryan E.A, Malykh M.D., Sevastianov L.A., Yu Ying

Department of Applied Probability and Informatics,
Peoples' Friendship University of Russia
(RUDN University).

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Symbolic manipulation and numerical calculation

Symbolic manipulations and numerical calculations are two opposed approaches to solution of the same problems. However, many problems which we try to solve symbolically, in finite terms, have arisen many centuries ago. Numerical methods of the last centuries dictated their formulations.

Example

We study the compass-and-straightedge constructions but we don't use these devices in practice long ago.

Symbolic integration of ODEs

We speak that we integrate ODE in finite terms (or in symbolic/analytic form)

- if we find the symbolic expression for the solution in elementary functions (Liouvillean approach).
- if we can describe all partial solutions of ODE as ratio of two convergent power series (Fuchs and Painlevé approach).

Observation

The notion of symbolic integration contains anyway the reference to the numerical methods of the past centuries.

Ref.: Malykh M.D. // Journal of Mathematical Sciences, 209:6, 2015.

Simple illustration

Example

In 1897 Painlevé has shown that all partial solutions of ODE

$$\frac{d^2x}{dt^2} = 6x^2 + t \tag{1}$$

now called 1st Painlevé transcendents, are described as ratio of two convergent power series.

- In the XIX century, this circumstance was very impotent because in those days the power series were really used for the numerical integration of the differential equations.
- In the XXI century, "NIST Digital Library of Mathematical Functions" [`dlmf.nist.gov`, §32.17] recommends to use Runge-Kutta method for the numerical integration of (1).

Our goal

Modern method for integration of the system of the differential equations is finite differences method (FDM).

Idea

We believe that all transcendental functions can be reconsidered as solutions of such differential equations, for which we don't feel a difference between exact and approximate solutions.

In the present report, we would like to consider one of the most important class of such functions, namely, the elliptic functions.

Jacobi elliptic functions

Jacobi elliptic functions are the solution

$$p = \operatorname{sn} t, \quad q = \operatorname{cn} t, \quad r = \operatorname{dn} t$$

of nonlinear autonomous system

$$\begin{cases} \dot{p} = qr, \\ \dot{q} = -pr, \\ \dot{r} = -k^2 pq, \end{cases}$$

with initial condition

$$p = 0,\ q = r = 1 \text{ at } t = 0.$$

These functions can be represented everywhere as the ration of two power series.

Ref.: `dlmf.nist.gov`, § 22.

Notations

Consider the autonomous system of the differential equations

$$\frac{d\vec{x}}{dt} = F(\vec{x}), \quad F \in \mathbb{Q}[\vec{x}],$$

on the interval $0 \le t \le T$ with the initial conditions

$$\vec{x}|_{t=0} = \vec{x}_0.$$

We divide the interval $[0, T]$ into parts with the step Δt by points $t_1, \ldots t_{N-1}$ and take

$$t_0 = 0, \quad t_N = T.$$

Value of approximate solution at point $t = t_n$ is designated as $\vec{x}_n$ and value of exact solution is designated as $x(t_n)$.

Differential schemes

FDM suggests replacing the original system of differential equations with algebraic equations (scheme) of the form

$$F(x, \hat{x}, \Delta t) = 0, \quad F \in \mathbb{Q}[x, \hat{x}, \Delta t]$$

in commonly used notations. Here and bellow the arrows over letters are forget.
These equations defines algebraical correspondence between neighboring layers x and $\hat{x}$, which are usually investigated as points of two affine or projective spaces.

Ref.: Samarskii A.A. The Theory of Difference Schemes. Dekker: NY, 2001.

Difficulties of FDM

Since the differential scheme is the system of algebraical equations, we can conserve algebraical properties of the exact solution. However standard explicit schemes don't conserve algebraic integrals of motion.

Example

System

$$\begin{cases} \dot{p} = qr, \\ \dot{q} = -pr, \\ \dot{r} = -k^2 pq, \end{cases}$$

has two quadratic integrals

$$p^2 + q^2 = \text{const} \quad \text{and} \quad k^2 p^2 + r^2 = \text{const}$$

Standard scheme of Runge-Kutta (rk4) does not conserve them.

Total conservative differential schemes

Definition

The differential scheme

$$F(x, \hat{x}, \Delta t) = 0 \tag{2}$$

is called total conservative iff for any algebraical integral $u(x)$ the equation

$$u(\hat{x}) = u(x)$$

is the consequence of the system (2).

The equality is conserved precisely if transition from a layer to a layer becomes precisely, without rounding errors.

The implicit midpoint rule

$$\frac{dx}{dt} = F(x) \quad \Rightarrow \quad \frac{x_{n+1} - x_n}{\Delta t} = \frac{F(x_{n+1}) + F(x_n)}{2}$$

Theorem (Cooper, 1987)

The implicit midpoint rule automatically inherits each quadratic conservation law.

If the field of algebraical integrals of dynamic system is generated by quadratic forms, then the implicit midpoint rule is total conservative.

Ref.: *Sanz-Serna J.M.* // SIAM Review. 2016. Vol. 58, No. 1, pp. 3–33.

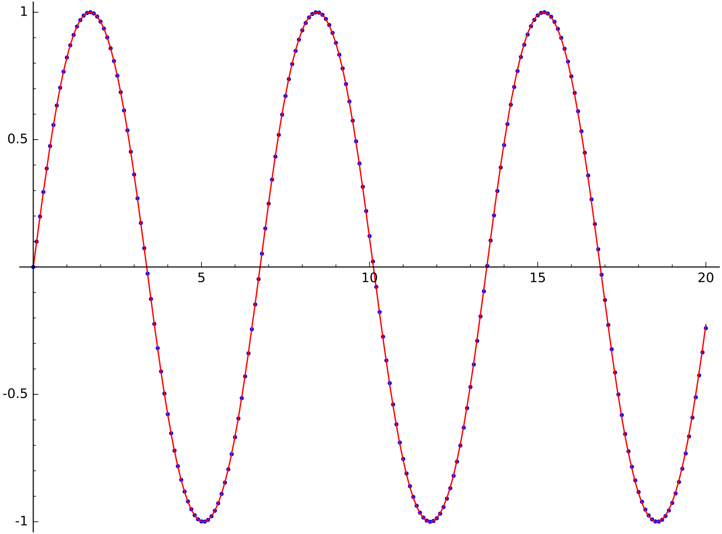
The implicit midpoint rule for elliptic functions

The implicit midpoint rule for Jacobi elliptic functions give the total conservative, but implicit differential scheme.
Yu Ying has executed a series of numerical experiments with this scheme in Sage. We can see that

- two algebraical integrals is conserved exactly,
- the error of rounding isn't a problem,
- the periodical nature is conserved approximately (or exactly?).

So we don't feel a difference between exact and approximate solutions.

$\mathrm{sn}\left(t, \frac{1}{2}\right),\ N = 1000$



The construction of explicit schemes

The implicit nature of the scheme is main difficulty for theoretical investigations and also for practical computations.

Problem

Given a system of differential equations

$$\dot{x} = F(x), \quad F \in \mathbb{Q}[x],$$

and a few integrals, construct an explicit differential scheme, exactly conserving the integrals of motion.

Here we give the solution for the case when the integrals of motion specify a curve in the projective space $\mathbb{P}^r$, where $\vec{x}$ varies.

Example

System

$$\begin{cases} \dot{p} = qr, \\ \dot{q} = -pr, \\ \dot{r} = -k^2 pq, \end{cases}$$

has two quadratic integrals

$$p^2 + q^2 = \text{const} \quad \text{and} \quad k^2 p^2 + r^2 = \text{const}$$

These integrals specify an elliptic curve in the space pqr. All layers coincide with this curve:

- exact solution define an automorphism of this curve.
- total conserve scheme also define an algebraic correspondence on this curve.

Elliptic case

The theory of algebraic curves give us the following.

Theorem

Any explicit total conservative difference scheme with elliptic layers defines birational automorphism and can be written as

$$\int_{x}^{\hat{x}} H dx_1 = \lambda(\Delta t).$$

where $H dx_1$ *is the differential form of the first kind.*

Example

The differential form of the first kind on the curve

$$p^2 + q^2 = 1 \quad \text{and} \quad k^2 p^2 + r^2 = 1$$

is equal to

$$\frac{dp}{qr},$$

thus explicit total conservative scheme (if it exists) can be written as

$$\int\limits_{(p,q,r)}^{(\hat{p},\hat{q},\hat{r})} \frac{dp}{qr} = \lambda(\Delta t).$$

Exact solution is described also as

$$\int\limits_{(p,q,r)}^{(\hat{p},\hat{q},\hat{r})} \frac{dp}{qr} = \Delta t.$$

Example: the difference scheme

By additions theorem for Jacobi functions we can write

$$\int_{(p,q,r)}^{(\hat{p},\hat{q},\hat{r})} \frac{dp}{qr} = \lambda(\Delta t)$$

in the algebraical form as

$$\begin{cases} \hat{p} = \dfrac{p \operatorname{cn}\lambda \operatorname{dn}\lambda + \operatorname{sn}\lambda qr}{1 - k^2 p^2 \operatorname{sn}^2 \lambda} \\ \hat{q} = \dfrac{q \operatorname{cn}\lambda - \operatorname{sn}\lambda \operatorname{dn}\lambda pr}{1 - k^2 p^2 \operatorname{sn}^2 \lambda} \\ \hat{r} = \dfrac{r \operatorname{dn}\lambda - k^2 \operatorname{sn}\lambda \operatorname{cn}\lambda pq}{1 - k^2 p^2 \operatorname{sn}^2 \lambda} \end{cases}$$

Thus $\operatorname{sn}\lambda$ has to be an algebraical function of Δt and hasn't to be equal to $\operatorname{sn}\Delta t$.

Example: approximation

The differential scheme approximates the differential equations with degree k iff

$$\lambda = \Delta t + \mathcal{O}(\Delta t^{k+1})$$

or

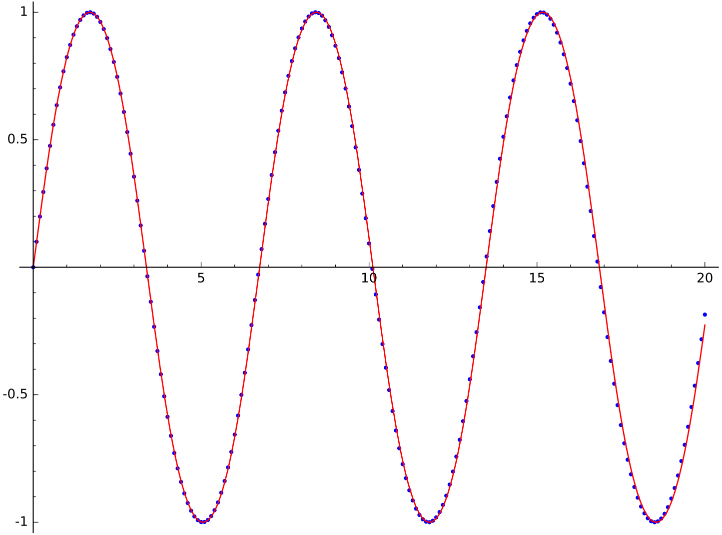
$$\operatorname{sn} \lambda = [\operatorname{sn} \Delta t]_k + \mathcal{O}(\Delta t^{k+1}),$$

where $[\ldots]_k$ designates the Taylor polynomial of degree k.
In particular, for $k = 1$

$$\operatorname{sn} \lambda = \Delta t, \quad \operatorname{cn} \lambda = \sqrt{1 - \Delta t^2}, \quad \operatorname{dn} \lambda = \sqrt{1 - k^2 \Delta t^2}$$

This differential scheme gives us exactly Gudermann's method for calculation of elliptic functions [Weierstrass, Bd. 1, Art. 1].

$\mathrm{sn}\left(t, \frac{1}{2}\right)$, $N = 200$, $k = 1$



Is it a differential scheme?

Gudermann scheme for calculation of elliptic function

$$\begin{cases} \hat{p} = \dfrac{p\sqrt{1-\Delta t^2}\sqrt{1-k^2\Delta t^2} + \Delta t q r}{1-k^2p^2\Delta t^2} \\ \hat{q} = \dfrac{q\sqrt{1-\Delta t^2} - \Delta t\sqrt{1-k^2\Delta t^2} p r}{1-k^2p^2\Delta t^2} \\ \hat{r} = \dfrac{r\sqrt{1-k^2\Delta t^2} - k^2\Delta t\sqrt{1-\Delta t^2} p q}{1-k^2p^2\Delta t^2} \end{cases}$$

is **almost** differential scheme, but

- its equations contain the radicals with respect to Δt,
- it is defined only on one layer

$$p^2 + q^2 = 1 \quad \text{and} \quad k^2p^2 + r^2 = 1,$$

not at all points of the space pqr.

Partial differential scheme

By analogy with notion of a partial solution of ODE we will accept the following.

Definition

Let $(m, \hat{m})$-correspondence be defined on a variety V enclosed in the protective space $\mathbb{P}^r$. If a point $x \in V$ correspondent one value $\hat{x}$, which can be described by Puisex series

$$\hat{x} = x + f(x)\Delta t + \dots,$$

then we can call that this correspondence is a partial differential scheme of type $(m, \hat{m})$ approximating the ODE $\dot{x} = f(x)$.

The variety V is called a layer, the dimension of $V \subset \mathbb{P}^r$ is called a dimension of the scheme and so on.

Classification of partial explicit schemes

The theory of algebraic correspondences [Zeuthen, 1914] gives the following.

Theorem

There are only two types of partial explicit schemes of dimension 1*:*

- *schemes with layers of genus 0,*
- *schemes with layers of genus 1*

All schemes with layers of genus 1 can be described by means of Abelian integral of the first kind. Thus all of them are birational $(m = \hat{m} = 1)$.
Gudermann scheme for elliptic function is a typical example of partial differential scheme with layers of genus 1.

Existence of total conservative explicit schemes

Any total conservative scheme is made of partial differential schemes.

Theorem

If the integrals of motion specify a curve of the genus $\rho > 0$ in the space $\mathbb{P}^r$, then total conservative explicit schemes does not exist.

In general, the curve of degree equal or more than 3 has genus $\rho > 0$, thus there aren't total conservative explicit schemes by purely geometric reasons.

What is elliptic functions?

Deduction

The autonomous system of ODEs, integrated in the elliptic functions, plays an especial role not only in the analytical theories, but also in FDM.

For this system we can write the excellent differential schemes:

- implicit total conservative $(5, 5)$-scheme, defined on all projective space pqr,
- explicit partial schemes, defined on the elliptic curves

$$p^2 + q^2 = C_1 \quad \text{and} \quad k^2p^2 + r^2 = C_2.$$

In general, the autonomous system with algebraical integrals can't be approximated by such differential schemes.

The end.

Calculations made in `SageMath version 7.5.1, Release Date: 2017-01-15`.
See additional materials on `http://malykhmd.neocities.org`.

Total periodic schemes

Approximate solution calculated by conservative differential scheme can be exactly periodic.

Definition

A partial differential scheme is called total periodic if there is the sequence

$$\{\Delta t_n \in \overline{\mathbb{Q}}\}$$

such that $x_n = x_0$, where $\{x_m\}$ is the approximate solution at $\Delta t = \Delta t_n$.

Here n is the number of points per period and $n\Delta t_n$ is the period of the approximate solution.

Theorem

If $n\Delta t_n \to T$, than the number T is the period of the exact solution.

The periodicity of our scheme

For Gudermann scheme we have

$$x_n = x_0 \quad \Rightarrow \quad n\lambda = \int_{x_0}^{x_0} \frac{dp}{qr} = 4K.$$

For scheme of 1st degree

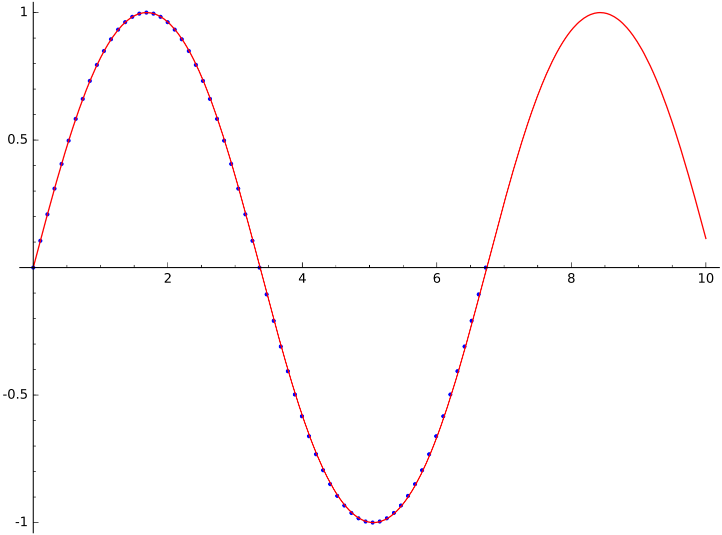
$$\operatorname{sn}\lambda = \Delta t.$$

Thus

$$\Delta t_n = \operatorname{sn}\frac{4K}{n} \in \overline{\mathbb{Q}}$$

and therefore our scheme is total periodic.

$\mathrm{sn}\left(t, \frac{1}{2}\right)$, $n = 2^6$



We have calculated Δt_n at $n = 2^s$ by formulas of a half corner.

The periodicity of our scheme

Approximate period is equal to

$$n\Delta t_n = n\,\mathrm{sn}\,\frac{4K}{n} = 4K - \frac{k^2+1}{6}\frac{4^3K^3}{n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)$$

Thus our difference scheme conserve exact the periodical nature of motion but we calculate the value of the period with small error

$$\frac{k^2+1}{6}\frac{4^3K^3}{n^2}.$$

The total end.

Calculations made in `SageMath version 7.5.1, Release Date: 2017-01-15`.
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