Elliptic functions and finite difference method

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April ??, 2018, ver. August 26, 2018

Symbolic integration and numerical calculation

Many problems which we try to solve in finite terms have arisen many centuries ago. Numerical methods of the last centuries dictated their formulations.

Example

We study the compass-and-straightedge constructions but we don't use these devices in practice long ago.

Why is differential Galois theory not widely used?

This is a typical answer from `mathoverflow.net`:

> *Differential Galois theory is concerned with linear differential equations in one variable. Specialists in differential equations tend to care about more than that (e.g., PDE, nonlinear equations), and the aspect of solvability addressed by differential Galois theory is largely of no interest to them even for linear differential equations. Even ordinary Galois theory is not really of much importance to analysts needing roots of a polynomial in* $\mathbb{C}$*. Nobody* **really** *cares about being able write roots in a very restrictive form related to solvable Galois groups. It is mostly of* **historic interest**.

Galois differential theory

In the studies of Galois differential theory, we use the concept of elementary functions.

- In the times of Liouville, when these studies were initiated, the functions have been considered elementary if their tables of values were available for common use.
- At present, this class of functions is much narrower than the set of functions, for which the computation algorithms are implemented in all systems of computer algebra. Thus now Galois differential theory 'is mostly of historic interest'.

Power series

In XIX century the power series really used for the numerical integration of the differential equations. Therefore mathematicians of the XIX century have allocated those functions which can be presented by power series.

Idea

Elementary functions and the higher transcendental functions are such solutions of autonomous system of the differential equations

$$\dot{\vec{x}} = f(\vec{x}), \quad f \in \mathbb{Q}[\vec{x}],$$

which can be calculated by means of power series at all values of variable t.

The elementary functions

The elementary functions (like $\exp t, \sin t, \cos t$) are such solutions which can be represented by power series everywhere.
The solutions of any system of the linear differential equations

$$\dot{\vec{x}} = f(\vec{x}), \quad f \in \mathbb{Q}[\vec{x}],$$

can be represented by power series which converges everywhere, but all these solutions are expressed by help of the solution of the scalar equation

$$\dot{x} = x,$$

known as $\exp t$.

The higher transcendental functions

The higher transcendental functions are such solutions of autonomous systems of the differential equations

$$\dot{\vec{x}} = f(\vec{x}), \quad f \in \mathbb{Q}[\vec{x}],$$

which can be represented as the ration of two power series

$$x_1 = \frac{a_0 + a_1 t + a_2 t^2 + \dots}{b_0 + b_1 t + b_2 t^2 + \dots},$$

which converge at all values of t.
This means that $\vec{x}$ is meromorphic functions of t and the system has Painlevé property.

Ex.: Jacobi elliptic functions

Jacobi elliptic functions are the solution

$$p = \operatorname{sn} t, \quad q = \operatorname{cn} t, \quad r = \operatorname{dn} t$$

of nonlinear system

$$\begin{cases} \dot{p} = qr, \\ \dot{q} = -pr, \\ \dot{r} = -k^2 pq, \end{cases}$$

with initial condition

$$p = 0,\ q = r = 1 \text{ at } t = 0.$$

These functions can be represented everywhere as the ration of two power series, for example,

$$\operatorname{sn} t = \frac{1}{\sqrt{k}} \frac{H(t)}{\Theta(t)}.$$

Symbolic integration in analytical theory of ODEs

In this theory, integration in finite terms means

- constructing of the algebraic substitution which reduces the initial system to a system with Painlevé property, or
- search the symbolic expression for the solution in elementary or higher transcendental functions.

Anyway we can see the reference to the computational techniques of the past centuries.

Historical notes

1820s Christoph Gudermann has made tables of hyperbolic and elliptic functions by help of power series.

1830s His pupil Karl Weierstrass has used power series for theoretical investigations of ODE, in particular in diploma-work has proved ‘Cauchy theorem’and meromorphy of elliptic functions.

Thus we can find roots of theory of analytical functions in the tables of transcendental functions.

Ref.: Weierstrass. Werke. Bd. 1, Abh. 1.

Modern numerical methods of integration

Standard method for integration of the autonomous system of the differential equations is finite differences method (FDM).

Idea

We believe that all transcendental functions can be reconsidered as solutions of such differential equations, for which the application of the finite difference method is particularly efficient.

In the present report, we would like to consider one of the most important class of such functions, namely, the elliptic functions.
Ref.: E.A. Ayryan, M.D. Malykh, L.A. Sevastianov. Finite differences method and integration of the differential equations in finite terms // Preprints of the Joint Institute for Nuclear Research (Dubna), 2018. No. P11-2018-17.

Notations

Consider the autonomous system of the differential equations

$$\frac{d\vec{x}}{dt} = F(\vec{x}), \quad F \in \mathbb{Q}[\vec{x}],$$

on the interval $0 \leq t \leq T$ with the initial conditions

$$\vec{x}|_{t=0} = \vec{x}_0.$$

We divide the interval $[0, T]$ into parts with the step Δt by points $t_1, \dots t_{N-1}$ and take

$$t_0 = 0, \quad t_N = T.$$

Value of approximate solution at point $t = t_n$ is designated as $\vec{x}_n$ and value of exact solution is designated as $x(t_n)$.

Differential schemes

FDM suggests replacing the original system of differential equations with an algebraic equations (scheme) of the form

$$F(x, \hat{x}; \Delta t) = 0$$

in commonly used notations, here and bellow the arrows over letters are forget.
This equation defines algebraical correspondence between neighboring layers x and $\hat{x}$, which are usually investigated as points on two affine or projective spaces.
Ref.: Samarskii A.A. The Theory of Difference Schemes. Dekker: NY, 2001.

Difficulties of FDM

As differential scheme is the system of algebraical equations, we can conserve algebraical properties of the exact solution. However standard explicit schemes don't conserve algebraic integrals of motion.

Example

System

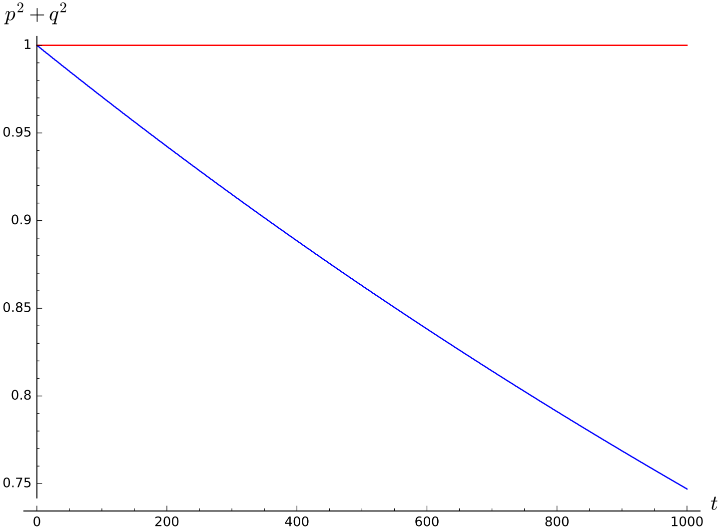
$$\begin{cases} \dot{p} = qr, \\ \dot{q} = -pr, \\ \dot{r} = -k^2 pq, \end{cases}$$

has two quadratic integrals

$$p^2 + q^2 = \text{const} \quad \text{and} \quad k^2 p^2 + r^2 = \text{const}$$

Standard for Sage scheme of Runge-Kutta (rk4) don't conserve them.

Standard rk4 for $k = \frac{1}{2}$, $\Delta t = \frac{1}{2}$



The periodic mouton

Jacobi elliptic functions have two periods. First is real number

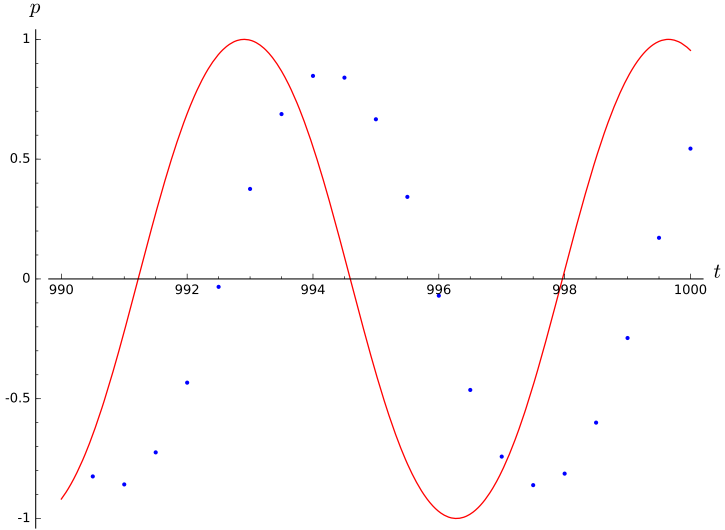
$$\omega = 4K(k^2) = 4\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{4\pi}{\operatorname{agm}(1, \sqrt{1-k^2})}$$

In particular for $k = 1/2$

```
sage: elliptic_kc(1/4).n()
1.68575035481260
```

This periodic nature of the mouton is lost.

Standard rk4 for $k = \frac{1}{2}$, $\Delta t = \frac{1}{2}$



Total conservative differential schemes

Definition

The differential scheme

$$F(x, \hat{x}; \Delta t) = 0 \tag{1}$$

is called total conservative iff for any algebraical integral $u(x)$ the equation

$$u(\hat{x}) = u(x)$$

is the consequence of the system (1).

The equality is conserved precisely if transition from a layer to a layer becomes precisely, without rounding errors.

The implicit midpoint rule

$$\frac{dx}{dt} = F(x) \quad \Rightarrow \quad \frac{x_{n+1} - x_n}{\Delta t} = \frac{F(x_{n+1}) + F(x_n)}{2}$$

Theorem (Cooper, 1987)

The implicit midpoint rule automatically inherits each quadratic conservation law.

If the field of algebraical integrals of dynamic system is generated by quadratic forms, then the implicit midpoint rule is total conservative.

Ref.: *Sanz-Serna J.M.* // SIAM Review. 2016. Vol. 58, No. 1, pp. 3–33.

The implicit midpoint rule for elliptic functions

The implicit midpoint rule for Jacobi elliptic functions give the differential scheme

$$\begin{cases} 4(p_{n+1} - p_n) = (q_{n+1} + q_n)(r_{n+1} + r_n)\Delta t \\ 4(q_{n+1} - q_n) = -(p_{n+1} + p_n)(r_{n+1} + r_n)\Delta t \\ 4(r_{n+1} - r_n) = -k^2(p_{n+1} + p_n)(q_{n+1} + q_n)\Delta t \end{cases} \tag{2}$$

This scheme

- is total conservative, but
- is not explicit.

Yu Ying has executed a series of numerical experiments with this scheme in Sage.

The organization of calculations by the implicit scheme

The elimination methods based on Gröbner's bases allow to organize calculations.

- 1st stage: symbolical preparation of the scheme. By elimination ideal technique we can find consequences of the form

  $$P(\hat{p}, p, q, r, \Delta t) = 0,\ Q(\hat{q}, p, q, r, \Delta t) = 0,\ R(\hat{r}, p, q, r, \Delta t) = 0.$$

  For implicit midpoint rule the polynomial P has the fifth degree with respect to $\hat{p}$.
- 2st stage: floating-point calculations in cycle. For transition to the following layer we solve the three **univariable** equations of the 5th degree.

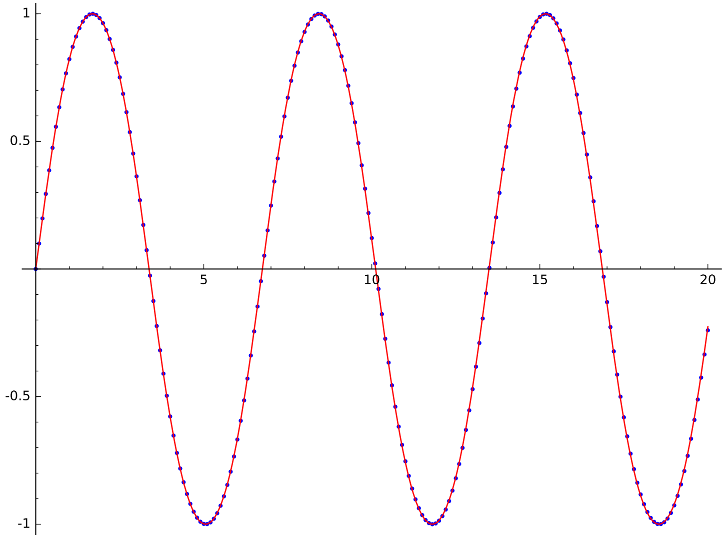
1st stage: searching of the polynomial P

```
sage: vars=var('p,q,r,pp,qq,rr,dt,k')
sage: K=QQ[vars]
sage: eqs=[4*(pp-p)-(q+qq)*(r+rr)*dt, 4*(qq
    -q)+(p+pp)*(r+rr)*dt, 4*(rr-r)+k^2*(q+qq
    )*(p+pp)*dt]
sage: J=K*eqs
```

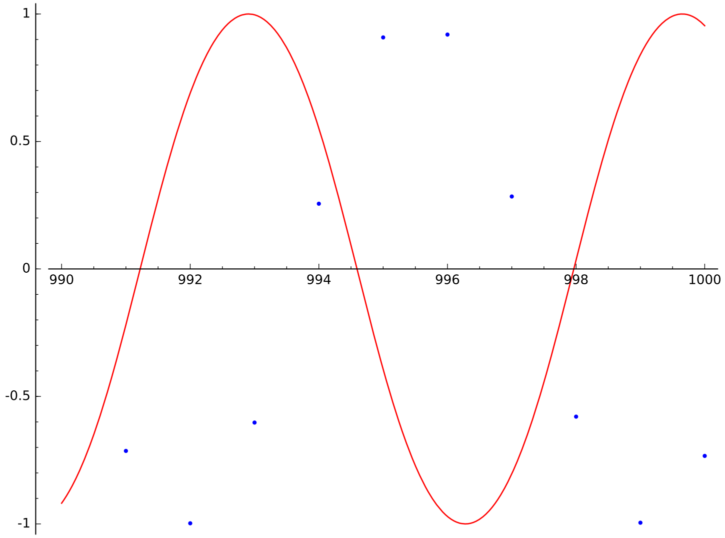
1st stage: the polynomial P

```
sage: J.elimination_ideal([K(qq),K(rr)])
Ideal (p^5*dt^4*k^4 + 3*p^4*pp*dt^4*k^4 +
    2*p^3*pp^2*dt^4*k^4 - 2*p^2*pp^3*dt^4*k
    ^4 - 3*p*pp^4*dt^4*k^4 - pp^5*dt^4*k^4 +
     16*p^2*q*r*dt^3*k^2 + 32*p*q*r*pp*dt^3*
    k^2 + 16*q*r*pp^2*dt^3*k^2 - 32*p^3*dt
    ^2*k^2 - 64*p*q^2*dt^2*k^2 - 32*p^2*pp*
    dt^2*k^2 - 64*q^2*pp*dt^2*k^2 + 32*p*pp
    ^2*dt^2*k^2 + 32*pp^3*dt^2*k^2 - 64*p*r
    ^2*dt^2 - 64*r^2*pp*dt^2 + 256*q*r*dt +
    256*p - 256*pp) of Multivariate
    Polynomial Ring in p, q, r, pp, qq, rr,
    dt, k over Rational Field
```

$\mathrm{sn}\left(t, \frac{1}{2}\right)$, $N = 200$



$\mathrm{sn}\left(t, \frac{1}{2}\right)$, $N = 1000$



Results

We can see that

- two algebraical integrals is conserved,
- the error of rounding isn't a problem,
- the periodical nature is conserved.

The implicit nature of the scheme is main difficulty for theoretical investigations and also for practical computations.

The construction of explicit schemes

Problem

Given a system of differential equations

$$\dot{x} = F(x), \quad F \in \mathbb{Q}[x],$$

and a few integrals, construct an explicit difference scheme, exactly conserving the integrals of motion.

Here we give the solution for the case when the integrals of motion specify a curve in the space $\mathbb{V}$, where $\vec{x}$ varies.

Example

System

$$\begin{cases} \dot{p} = qr, \\ \dot{q} = -pr, \\ \dot{r} = -k^2 pq, \end{cases}$$

has two quadratic integrals

$$p^2 + q^2 = \text{const} \quad \text{and} \quad k^2 p^2 + r^2 = \text{const}$$

These integrals specify an elliptic curve in the space $Opqr$. All layers coincide with this curve:

- exact solution define an automorphism of this curve.
- total conserve scheme also define an algebraic correspondence on this curve.

Existence of total conservative explicit schemes

Theorem

If the integrals of motion specify a curve of the genus $\rho > 1$ in the space $\mathbb{V}$, than total conservative explicit schemes does not exist.

In general, the curve of degree equal or more than 4 has genus $\rho > 1$, thus there aren't total conservative explicit schemes by purely geometric reasons.

Zeuthen theorem

Let there is an algebraic correspondence between curves C_1 and C_2,

- general point P_1 at the curve C_1 corresponds α_2 various points at the curve C_2 and general point P_2 at the curve C_2 corresponds α_1 various points at the curve C_1,
- η_1 is the number of the coincidences of two points P_1 corresponding to a point P_2 at the curve C_2
- ρ_i is the genus of the curve C_i.

These numbers are connected by Zeuthen formulæ

$$\eta_2 - \eta_1 = 2\alpha_1(\rho_2 - 1) - 2\alpha_2(\rho_2 - 1),$$

Ref.: Zeuthen. Lehrbuch der abzählenden Methoden der Geometrie, 1914. §65.

Proof of the theorem-1

If the integrals of motion specify a curve C in the space $\mathbb{V}$, any total conservative difference scheme defines algebraic correspondence between layers $C_1 = C$ and $C_2 = C$.
This scheme is explicit iff general point P_1 at first layer corresponds one point at second layer, so

$$\alpha_2 = 1, \quad \beta_2 = 0.$$

Thus by Zuethen formulæ

$$-\eta_1 = 2(\alpha_1 - 1)(\rho - 1),$$

what is possible only if $\alpha_1 = 1$.
This means that the correspondence is birational automorphism.

Proof of the theorem-2

Difference scheme has natural parameter Δt, thus it defines the one-parameter family of birational automorphisms.
It is possible, because by Hurwitz theorem the number N of all birational automorphisms of a curve with genus $\rho > 1$ is finite.

Ref.: Eslam E. Badr and Mohammed A. Saleem. Cyclic Automorphisms groups of genus 10 non-hyperelliptic curves, arXiv:1307.1254, 2013.

Two cases when the scheme may exist

The are two cases when the system

$$\dot{x} = F(x), \quad F \in \mathbb{Q}[x],$$

may be discretized by explicit total conservative scheme:

- if integrals specify an elliptic curve (genus is equal to 1), and
- if integrals specify an unicursal curve (genus is equal to 0)

Our example with Jacobi elliptic functions belong to first case.

Elliptic case

Theorem

Any explicit total conservative difference scheme with elliptic layers defines birational automorphism and can be written as

$$\int_{x}^{\hat{x}} H dx_1 = \lambda(\Delta t).$$

where $H dx_1$ is differential form of the first kind.

Sketch of proof. For explicit scheme, $\hat{x}$ is a rational function of x,

$$\int_{o}^{\hat{x}} H dx_1 = \alpha(\Delta t) \cdot \int_{o}^{x} H dx_1 + \beta(\Delta t).$$

Here α is an algebraic function without singularities, that is a constant. At $\Delta t = 0$ the automorphism is identity thus $\alpha = 1$.

Example

The differential form of the first kind on the curve

$$p^2 + q^2 = \text{const} \quad \text{and} \quad k^2 p^2 + r^2 = \text{const}$$

is equal to

$$\frac{dp}{qr},$$

thus explicit total conservative scheme (if it exists) can be written as

$$\int\limits_{(p,q,r)}^{(\hat{p},\hat{q},\hat{r})} \frac{dp}{qr} = \lambda(\Delta t).$$

Exact solution is described also as

$$\int\limits_{(p,q,r)}^{(\hat{p},\hat{q},\hat{r})} \frac{dp}{qr} = \Delta t.$$

Example: the difference scheme

By additions theorem for Jacobi functions we can write

$$\int_{(p,q,r)}^{(\hat{p},\hat{q},\hat{r})} \frac{dp}{qr} = \lambda(\Delta t)$$

in algebraical form

$$\begin{cases} \hat{p} = \dfrac{p \operatorname{cn} \lambda \operatorname{dn} \lambda - \operatorname{sn} \lambda qr}{1 - k^2 p^2 \operatorname{sn}^2 \lambda} \\ \hat{q} = \dfrac{q \operatorname{cn} \lambda - \operatorname{sn} \lambda \operatorname{dn} \lambda pr}{1 - k^2 p^2 \operatorname{sn}^2 \lambda} \\ \hat{r} = \dfrac{r \operatorname{dn} \lambda - k^2 \operatorname{sn} \lambda \operatorname{cn} \lambda pq}{1 - k^2 p^2 \operatorname{sn}^2 \lambda} \end{cases}$$

Thus $\operatorname{sn} \lambda$ has to be an algebraical function of Δt and hasn't to be equal to $\operatorname{sn} \Delta t$.

Example: approximation

The difference scheme approximate the differential equations with degree k iff

$$\lambda = \Delta t + \mathcal{O}(\Delta t^{k+1})$$

or

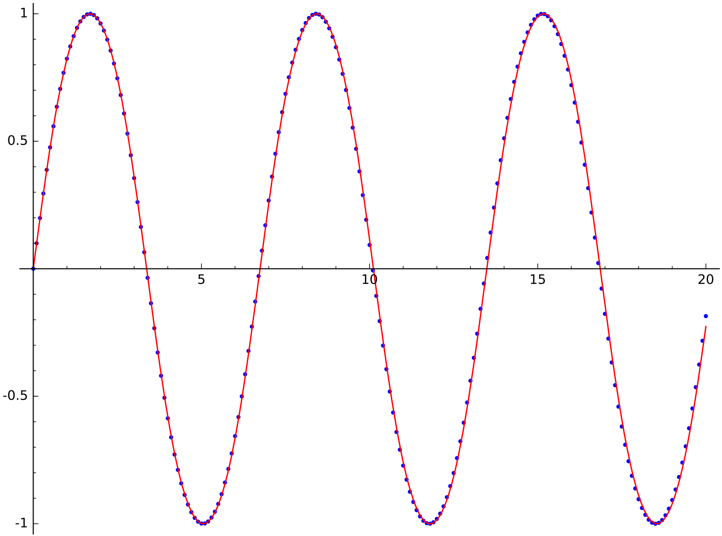
$$\operatorname{sn}\lambda = [\operatorname{sn}\Delta t]_k + \mathcal{O}(\Delta t^{k+1}),$$

where $[\dots]_k$ designates the Taylor polynomial of degree k.
In particular, for $k = 1$

$$\operatorname{sn}\lambda = \Delta t, \quad \operatorname{cn}\lambda = \sqrt{1 - \Delta t^2}, \quad \operatorname{dn}\lambda = \sqrt{1 - k^2 \Delta t^2}$$

This difference scheme give us exactly Gudermann's method for calculation elliptic functions.

$\mathrm{sn}\left(t, \frac{1}{2}\right)$, $N = 200$, $k = 1$



Total periodic schemes

Definition

Difference scheme is called total periodic if there is the sequence

$$\{\Delta t_n \in \overline{\mathbb{Q}}\}$$

such that $x_n = x_0$, where $\{x_m\}$ is approximate solution at $\Delta t = \Delta t_n$.

Here n is the number of points per period and $n\Delta t_n$ is a period of the approximate solution.

Theorem

If

$$n\Delta t_n \to T,$$

than the number T is a period of the exact solution.

The periodicity of our scheme

For explicit total conservative scheme we have

$$x_n = x_0 \quad \Rightarrow \quad n\lambda = \int_{x_0}^{x_0} H dx_1 = 4K.$$

For scheme of 1st degree

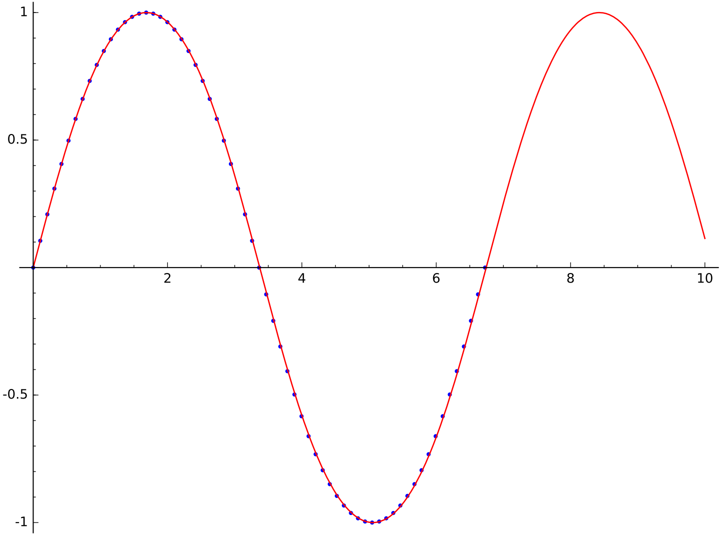
$$\operatorname{sn} \lambda = \Delta t.$$

Thus

$$\Delta t_n = \operatorname{sn} \frac{4K}{n} \in \overline{\mathbb{Q}}$$

and therefore our scheme is total periodic.

$\mathrm{sn}\left(t, \frac{1}{2}\right), n = 2^6$



We have calculated Δt_n at $n = 2^s$ by formulas of a half corner.

The periodicity of our scheme

Approximate period is equal to

$$n\Delta t_n = n\,\mathrm{sn}\,\frac{4K}{n} = 4K - \frac{k^2+1}{6}\frac{4^3K^3}{n^2} + \mathcal{O}\left(\frac{1}{n^4}\right)$$

Thus our difference scheme conserve exact the periodical nature of motion but we calculate the value of the period with small error

$$\frac{k^2+1}{6}\frac{4^3K^3}{n^2}.$$

What is elliptic functions?

The elliptic functions can be reconsidered as solutions of such differential equations, for which we can write the very good difference scheme. This scheme is

- explicit, that is calculations don't require the solution of the nonlinear equations,
- total conservative, that is all algebraical integrals of motion is conserved exactly,
- total periodic, that is periodical nature of motion is conserved exactly and value of the period is conserved with small error.

In general the autonomous system with algebraical integrals can't be approximated by explicit total conservative difference scheme.

The end.

Calculations made in `SageMath version 7.5.1, Release Date: 2017-01-15`.
See additional materials on `http://malykhmd.neocities.org`.