# On electromagnetic fields in closed waveguides with inhomogeneous filling

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# Motivation

Optical fibers are often used in the fabrication of the communication systems[1].

So called next-generation systems are very sensitive to the optical properties of the multicore fibers. For modeling these fibers are placed inside conducting boxes.

This is why it is very impotent to solve mathematically and computationally the problem of modeling a closed waveguide with complex filling described by piecewise constant functions $\varepsilon$ and $\mu$.

[1] Coffey V. C. Novel fibers use space to extend capacity limits. Photonics Spectra, 4:7 (2013).

# Waveguide

Consider a waveguide having:

- the constant cross-section $S$,
- ideally conducting walls,
- the filling which does not change along its axis and is described by the piecewise constant functions $\varepsilon$ and $\mu$ on the waveguide cross section, discontinuous on the line $\Gamma \subset S$.

Let the axis $Oz$ of Cartesian coordinates be directed along the waveguide axis.

# The electromagnetic field

The electromagnetic field is described by the vector fields $\vec{E}, \vec{H}$ smooth with their derivatives with respect to $z$ and $t$ on the cylinder $(S - \Gamma) \times Z \times T$. As well $\vec{E}$ and $\vec{H}$ satisfy

- Maxwell equations

$$\begin{cases} \operatorname{curl} \vec{E} = -\partial_t \frac{\mu}{c} \vec{H}, & \operatorname{div} \varepsilon \vec{E} = 0, \\ \operatorname{curl} \vec{H} = +\partial_t \frac{\varepsilon}{c} \vec{E}, & \operatorname{div} \mu \vec{H} = 0 \end{cases} \tag{1}$$

  inside the waveguide $S \times Z \times T$,

# The electromagnetic field, slide no. 2

- ideal conductivity conditions on waveguide walls

$$\vec{E} \times \vec{n} = 0, \quad \vec{H} \cdot \vec{n} = 0 \tag{2}$$

at the regular points of the boundary $\partial S \times Z \times T$,

- matching conditions

$$\begin{cases} [\vec{E} \times \vec{n}] = \vec{0}, & [\varepsilon \vec{E} \cdot \vec{n}] = 0 \\ [\vec{H} \times \vec{n}] = \vec{0}, & [\mu \vec{H} \cdot \vec{n}] = 0 \end{cases} \tag{3}$$

at regular points of the filling discontinuity boundary $\Gamma \times Z \times T$.

# Some fields

- If $E_z$ is equal to zero than the field is called transverse electric (TE-field).
- If $H_z$ is equal to zero than the field is called transverse electric (TM-field).
- If $\vec{E}$ and $\vec{H}$ have the form

$$\vec{E} = \vec{E}(x, y, z)e^{-i\omega t}, \quad \vec{H} = \vec{H}(x, y, z)e^{-i\omega t}$$

  than the field is called monochromatic with frequency $\omega$.
- If moreover the field have the form

$$\vec{E} = \vec{E}(x, y)e^{i\gamma z - i\omega t}, \quad \vec{H} = \vec{H}(x, y)e^{i\gamma z - i\omega t}$$

  than the field is called a normal wave. We can use the notations:

$$k = \frac{\omega}{c}, \quad \beta = \frac{\gamma}{k}.$$

# Classical results

**Theorem (A.N. Tikhonov and A.A. Samarskii, 1948)**

*In a hollow closed waveguide ($\varepsilon, \mu$ are constants) it is possible to introduce two scalar potentials, using which the Maxwell equations are reduced to a pair of uncoupled wave equations.*

The situation with the inhomogeneous filling is rather complicated and more difficult to solve.

# Hollow and filled waveguides

- For the hollow waveguides, the resulting problems are scalar, and one can use the well-developed methods, equally applicable to acoustics and quantum mechanics.
- For the waveguides with the inhomogeneous filling, one has to solve numerically the full vector electrodynamic problem.

This fact forces to work in exotic functional spaces and to use the finite elements method which is difficult to implement [Delitsyn, 1999; Lezar E., Davidson D.B., 2011].

# Normal modes

The most important consequence of Tikhonov-Samarskii theorem is the completeness of the normal waves system in a hollow waveguide.

**Theorem**

*Any wave propagating through the hollow closed waveguide can be presented as a superposition of TE and TM waves.*

In 1990s this consequence was generalized for the case of a waveguide, in which the filling varies over the transverse section, and is constant along the waveguide axis in terms of exotic functional spaces, what is difficult to implement on computer [Delitsyn, 2011].

# Our goals

1. We can avoid using of exotic functional spaces if instead of discontinuous cross components of the electromagnetic field $\vec{E}$ and $\vec{H}$ we can use continuous potentials. For hollow waveguides we can use two potentials but in general case we propose to use **four** potentials.
2. We believe that the theorem of field representation using potentials became shadowed by its consequence. We want to show that in terms of four potentials **the Maxwell equations are reduced to a pair of almost wave equations** (when $\varepsilon$ and $\mu$ are piecewise constant functions).
3. We show that these equations can be solved with the help of standard FEA softwares like FreeFem++.

# Helmholtz decomposition

Let us define the relation between the fields and potentials as

$$\vec{E}_\perp = \nabla u_e + \frac{1}{\varepsilon}\nabla' v_e, \quad \vec{H}_\perp = \nabla v_h + \frac{1}{\mu}\nabla' u_h. \tag{4}$$

Here we assume for brevity that

$\vec{A}_\perp = (A_x, A_y, 0)^T$ and $\nabla = (\partial_x, \partial_y, 0)^T, \quad \nabla' = (-\partial_y, \partial_x, 0)^T.$

**Theorem (Malykh M.D., Sevastianov L.A., 2018)**

*For any electromagnetic field $\vec{E}, \vec{H}$ in the closed waveguide, one can find such functions $u_e, u_h$ of the variables $z, t$ taking the values in the Sobolev space $\overset{o}{W}{}^1_2(S)$ and such functions $v_e, v_h$ of the variables $z, t$ taking the values in the Sobolev space $W^1_2(S)$, that the equality (4) is valid. The above representation is unique up to additive constants.*

# Helmholtz decomposition

While substituting the variables $\vec{E}$, $\vec{H}$ by four potentials and two components $E_z$, $H_z$ using the Eqs. (4) no solutions of Maxwell equations are lost. The conditions

$$u_e,\, u_h,\, E_z \in \overset{0}{W}{}_2^1(S) \quad \text{and} \quad v_e,\, v_h,\, H_z \in W_2^1(S)$$

replace the conditions at the filling discontinuities and the boundary conditions.

In terms of four potentials, the system of Maxwell equations has been separated into two independent systems.

**Theorem (Malykh M.D., Sevastianov L.A., 2018)**

*Any electromagnetic field $\vec{E}, \vec{H}$ is a superposition of TE and TM fields.*

# TM normal waves

Let's to face the spectral problem of our waveguide.
The TM normal wave is described by the potentials of the form

$$u_e = \tilde{u}_e e^{i\gamma z - i\omega t}, \quad u_h = \tilde{u}_h e^{i\gamma z - i\omega t}.$$

The amplitudes $\tilde{u}_e,\ \tilde{u}_h \in \overset{\circ}{W}{}^1_2(S)$ satisfy the equations in weak form

$$\begin{cases} \iint\limits_S \varepsilon(\nabla u, \nabla \tilde{u}_e) dxdy = -\gamma^2 \iint\limits_S \varepsilon u \tilde{u}_e dxdy + k\gamma \iint\limits_S \varepsilon u \tilde{u}_h dxdy, \\ \iint\limits_S \frac{1}{\mu}(\nabla u, \nabla \tilde{u}_h) dxdy = -k\gamma \iint\limits_S \varepsilon u \tilde{u}_e dxdy + k^2 \iint\limits_S \varepsilon u \tilde{u}_h dxdy, \end{cases}$$

for any $u \in \overset{\circ}{W}{}^1_2(S)$.

# Eqs. in the operator form

We may rewrite the system of weak equations in operator form, using the standard technique of the Sobolev spaces theory.

$$\begin{cases} A_{\varepsilon}\tilde{u}_e = -\gamma^2 B_{\varepsilon}\tilde{u}_e + k\gamma B_{\varepsilon}\tilde{u}_h, \\ A_{\frac{1}{\mu}}\tilde{u}_h = -k\gamma B_{\varepsilon}\tilde{u}_e + k^2 B_{\varepsilon}\tilde{u}_h \end{cases} \tag{5}$$

Here $A_{\varepsilon}$ and $A_{\frac{1}{\mu}}$ are bounded self-adjoint operators and $B_{\varepsilon}$ is a compact self-adjoint operator.

**Definition**

All the points of the $k\gamma$-plane where this problem has nontrivial solution form **the dispersive curve** of the waveguide.

# Almost Helmholtz equation

Excluding $\tilde{u}_h$ from this system, we get from (5) the next one

$$A_\varepsilon \tilde{u}_e = -\gamma^2 \left( B_\varepsilon + B_\varepsilon \left( \frac{1}{k^2} A_{\frac{1}{\mu}} - B_\varepsilon \right)^{-1} B_\varepsilon \right) \tilde{u}_e.$$

**Theorem**

*Any monochromatic electromagnetic TM field can be represented as a superposition of normal TM waves.*

Analogues theorem is valid for TE-field also.
The completeness of the system of normal modes was established in the works of A.N. Bogolyubov, A.L. Delitsyn, Yu.G. Smirnov, A.G. Sveshnikov[2].

[2] See the review by A.L. Deltsyn in Comp. math., 2011

# Solving of the spectral problem

For the calculation we will apply the truncation method: we will

- use finite element space $V_h$ instead of Sobolev space $\overset{0}{W}{}_2^1(S)$ and
- change the operators $A_\varepsilon, A_{\frac{1}{\mu}}$ and $B_\varepsilon$ to the sparse matrices, generated by the same bilinear forms.

For calculation of these matrices and further manipulations with block-sparse matrices we use free FEA software FreeFem++[3].

[3] Created by Université Pierre et Marie Curie and Laboratoire Jacques-Louis Lions, since 1987

# The program for the construction of dispersion curves

We have written the FreeFem++ program for the construction of waveguide dispersion curves. We can work with

- any waveguide cross-sections if the boundaries can be described parametrically with the help of elementary functions,
- any piecewise-constant filling described with the help of algebraic inequalities.

# Reduction to standard eigenvalue problem

Substituting $\gamma = k\beta$ we can rewrite our eigenvalue problem (5) in the block-sparse form

$$\begin{pmatrix} A_\varepsilon & 0 \\ 0 & A_{\frac{1}{\mu}} \end{pmatrix} \begin{pmatrix} \tilde{u}_e \\ \tilde{u}_h \end{pmatrix} = k^2 \begin{pmatrix} -\beta^2 B_\varepsilon & \beta B_\varepsilon \\ -\beta B_\varepsilon & B_\varepsilon \end{pmatrix} \begin{pmatrix} \tilde{u}_e \\ \tilde{u}_h \end{pmatrix}$$

With $k^2$ as eigenvalue and $\beta$ as a parameter we get a standard algebraic eigenvalue problem

$$Au = \lambda Bu.$$

Here $A, B$ are sparse matrices.

# The simple test

The dispersive curve of the hollow waveguide with $S = 1 \times 1$ can be described analytically by the equation

$$\pi^2(n^2 + m^2) = \varepsilon\mu k^2 - \beta^2 k^2, \quad n, m \in \mathbb{Z},$$

Thus at $\varepsilon = \mu = 1$ и $\beta = 0.5$ we have exact values

$$k_1^2 = 26.3189450695716 < k_2^2 = k_3^2 = 65.7973626739290 < \dots$$

Using the mesh with 2120 triangles we get

$$k_1^2 = 26.360 < k_2^2 = 66.069 < k_3^2 = 66.058,$$

dividing the mesh we get

$$k_1^2 = 26.329 < k_2^2 = 65.863 < k_3^2 = 65.864.$$

So we can see the convergence in numerical experiments.

# The numerical example

Construct the dispersive curve of the waveguide with the cross-section

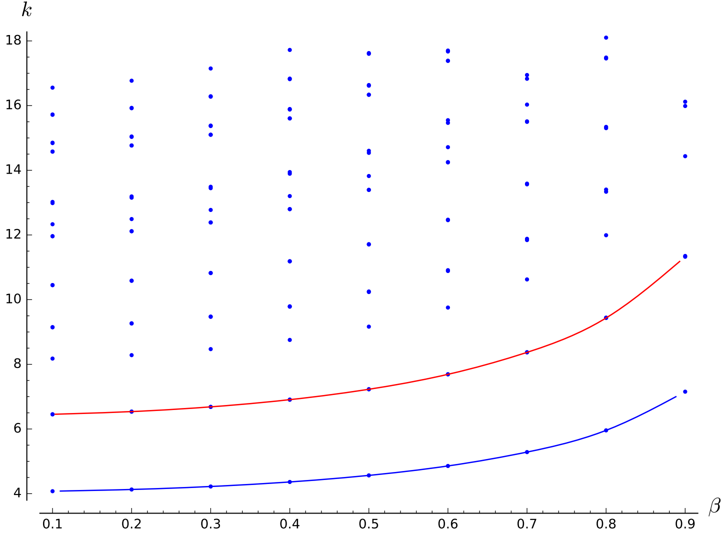
$$S = \{0 < x < 1\} \times \{0 < y < 1\},$$

the piecewise-constant filling is

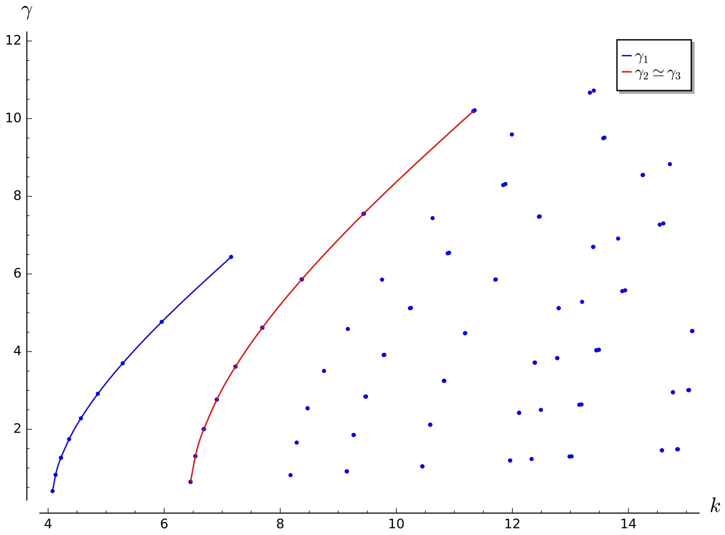
$$\varepsilon = \begin{cases} 1.2, & (x-0.5)^2 + (y-0.3)^2 < 0.5 \\ 1, & \text{otherwise} \end{cases}, \quad \mu = 1.$$

We can use the mesh with 2120 triangles and solve the eigenvalue problem at several values of $\beta$ (the step $\Delta\beta = 0.1$).

# The eigenvalues at several values of $\beta$.



# The dispersive curve.



# Summary

We have considered the closed waveguide with an arbitrary cross-section $S$ with piecewise-constant $\varepsilon$ and $\mu$.

**Main idea**

Instead of discontinuous cross components of the electromagnetic field $\vec{E}$ and $\vec{H}$ we propose to use four potentials $u_e, u_h \in \overset{\circ}{W}{}_2^1(S)$ and $v_e, v_h \in W_2^1(S)$.

We used this idea

- to generalize the Thikhonov-Samarskii theory for hollow waveguides,
- to create the software for calculation of dispersive curves.

# Thank you for attention.

# Eqs. for $u_e, u_h$

From Maxwell equations it follows that the potentials $u_e$, $u_h$ and $E_z$ are elements of $\overset{0}{W}{}_2^1(S)$, coupled by the equations

$$\begin{cases} \iint\limits_S \varepsilon(\nabla u, \nabla u_e) dx dy = \partial_z \iint\limits_S \varepsilon u E_z dx dy, \\ \iint\limits_S \frac{c}{\mu}(\nabla u, \nabla u_h) dx dy = -\partial_t \iint\limits_S \varepsilon u E_z dx dy, \end{cases} \tag{6}$$

for any $u$ from $C_0^\infty(S)$, where

$$E_z = \partial_z u_e + \partial_t u_h.$$

# Eqs. for $v_e, v_h$

From Maxwell equations it follows that the potentials $v_e$, $v_h$ и $H_z$ are elements of $W_2^1(S)$, coupled by the equations

$$\begin{cases} \iint\limits_S \frac{c}{\varepsilon}(\nabla v, \nabla v_e)dxdy = \partial_t \iint\limits_S \mu v H_z dxdy, \\ \iint\limits_S \mu(\nabla v, \nabla v_h)dxdy = \partial_z \iint\limits_S \mu v H_z dxdy, \end{cases} \tag{7}$$

for any $v$ from $C^\infty(S)$, where

$$H_z = \partial_z v_h - \partial_t v_e.$$