Calculation of Abelian integrals in computer algebra systems

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Integration of algebraic functions in CAS

Symbolic integration of elementary functions is oldest problem, partial solved in 1960s in Maxima.
Main unsolved part of the problem is the integration of algebraic functions (calculation of Abelian integrals).
Let's look as modern CAS cope with test examples.

Ex 1. Elliptic integral of 1st kind

The integral

$$\int \frac{dx}{\sqrt{1-x^3}}$$

or

$$\int \frac{dx}{y}$$

on the elliptic curve

$$y^2 = 1 - x^3$$

is the Abelian integral of 1st kind, that is the integral without any non-integrable singularities.
Thus this integral has two independent periods and is not elementary function.

Ex. 1. Wolfram Alpha

int(1/sqrt(1-x^3),x)

≡ Browse Examples

Indefinite integral:

$$\int \frac{1}{\sqrt{1-x^3}}\,dx = \frac{2\,i\,\sqrt{(-1)^{5/6}(x-1)}\,\sqrt{x^2+x+1}\,F\left(\sin^{-1}\left(\frac{\sqrt{-i\,x-(-1)^{5/6}}}{\sqrt[4]{3}}\right)\middle|\,\sqrt[3]{-1}\right)}{\sqrt[4]{3}\,\sqrt{1-x^3}} + \text{const}$$

$\sin^{-1}(x)$ is the inverse si

$F(x\,|\,m)$ is the elliptic integral of the first kind with parame

Ex. 1. Maple

```
> int(1/sqrt(1-x^3), x)
```

$$\frac{1}{\sqrt{-x^3+1}}\left(-\frac{2\,\mathrm{I}}{3}\sqrt{3}\sqrt{\mathrm{I}\left(x+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)\sqrt{3}}\sqrt{\frac{x-1}{-\frac{3}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}}\sqrt{-\mathrm{I}\left(x+\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)\sqrt{3}}\;\mathrm{EllipticF}\left(\frac{\sqrt{3}\sqrt{\mathrm{I}\left(x+\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)\sqrt{3}}}{3},\sqrt{\frac{\mathrm{I}\sqrt{3}}{-\frac{3}{2}+\frac{\mathrm{I}\sqrt{3}}{2}}}\right)\right)$$

Summary: Both systems don't recognize the type of the integral and reduce it to integral

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

What for?

Ex. 2. Elliptic integral of 2st kind

The integral

$$\int \frac{xdx}{\sqrt{x^3 + x + 1}}$$

or

$$\int \frac{dx}{y}$$

on the elliptic curve

$$y^2 = x^3 + x + 1$$

is the Abelian integral of 2st kind, that is the integral with one pole of 2nd order.

Thus this integral has two independent periods and is not elementary function.

Ex. 2. Wolfram Alpha

int(x/sqrt(x^3+x+1),x)

Browse Examples

Indefinite integral:

$$\int \frac{x}{\sqrt{x^3+x+1}}\,dx =$$

$-\Big(\big(2\,(x -$ root of x^3+x+1 near $x = 0.341164 - 1.16154\,i$ $)$

$\sqrt{((}$ root of x^3+x+1 near $x = -0.682328$ $- x)/$

$($ root of x^3+x+1 near $x = -0.682328$ $-$

root of x^3+x+1 near $x = 0.341164 + 1.1$

$\sqrt{((-x+}$ root of x^3+x+1 near $x = 0.341164 + 1.16$

$($ root of x^3+x+1 near $x = 0.341164 + 1.161$

Ex. 2. Maple

> $int\left(\frac{x}{\mathrm{sqrt}(x^3+x+1)}, x\right)$

$$\frac{1}{\sqrt{x^3+x+1}}\left(\frac{2\,I}{3}\sqrt{3}\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6} - \frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)\right.$$

Ex. 2. Summary

Both systems don't recognize the type of the integral and reduce it to Legendre integrals 1st and 2nd kind, that are

$$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and} \quad \int \frac{x\,dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

The expression occupies 3 screens and contains a lot of numerical constants. What for? How we can use it?

Ex. 3. Algebraic integral

Problem

For given Abelian integral recognize that this integral is algebraic function.

Algorithms for solving of this problem were given interdependently by Jan Ptaszicki (1888) and Weiersrtass (1875).
For test we will use

$$\int \frac{(-3x^5+x+2)dx}{2(x+\sqrt{x^5+x+1})^2\sqrt{x^5+x+1}} = \frac{x}{x+\sqrt{x^5+x+1}} + C.$$

Ex. 3. Wolfram Alpha

int((-3*x^5+x+2)/(2*(x+sqrt(x^5+x+1))^2*sqrt(x^5+x+1)),x)

Browse Examples Su

Indefinite integral: Step-by-ste

$$\int \frac{-3x^5+x+2}{2\left(x+\sqrt{x^5+x+1}\right)^2\sqrt{x^5+x+1}}\,dx = \frac{x\left(\sqrt{x^5+x+1}-x\right)}{x^5-x^2+x+1} + \text{consta}$$

Ex. 3. Maple

$$> u := simplify\left(diff\left(\frac{x}{x+\mathrm{sqrt}(x^5+x+1)}, x\right)\right)$$

$$u := \frac{-3x^5+x+2}{2\left(x+\sqrt{x^5+x+1}\right)^2\sqrt{x^5+x+1}}$$

$$> int(u, x)$$

$$\int \frac{-3x^5+x+2}{2\left(x+\sqrt{x^5+x+1}\right)^2\sqrt{x^5+x+1}}\,dx$$

Summary. Wolfram Alpha recognize algebraic integral and Maple can't. Existence of the button 'Step-by-step' on previous slide indicate that Wolfram Alpha use elementary heuristic.

Ex. 4. Elementary integral

Problem

For given Abelian integral recognize that this integral is elementary function.

An algorithm for solving of this problem was given by Jan Ptaszicki (1888). For elliptic integral this problem was solved by Chebyshev et al. It was very popular problem in Russian math. journals in the second half of the 19th century.

For test we will use

$$\int \frac{(5x^2 + 2\sqrt{x^3 + x + 1} + 1)dx}{2(x + \sqrt{x^5 + x + 1})\sqrt{x^5 + x + 1}} = \ln\left(x + \sqrt{x^5 + x + 1}\right) + C.$$

Ex. 4. Wolfram Alpha

```
int((5*x^4 + 2*sqrt(x^5 + x + 1) + 1)/(x^5 + sqrt(x^5 + x + 1)*x + x + 1),x)
```

Browse Examples

Input:

$$\int \frac{5x^4 + 2\sqrt{x^5 + x + 1} + 1}{x^5 + \sqrt{x^5 + x + 1}\, x + x + 1}\, dx$$

Indefinite integral:

(no result found in terms of standard mathematical functions)

Ex. 4. Maple

> $u := simplify(diff(\ln(x + \mathrm{sqrt}(x^5 + x + 1)), x))$

$$u := \frac{5x^4 + 2\sqrt{x^5+x+1} + 1}{\sqrt{x^5+x+1}\left(2x + 2\sqrt{x^5+x+1}\right)}$$

> $int(u, x)$

$$\int \frac{5x^4 + 2\sqrt{x^5+x+1} + 1}{\sqrt{x^5+x+1}\left(2x + 2\sqrt{x^5+x+1}\right)}\,\mathrm{d}x$$

Summary. Wolfram Alpha and Maple can't recognize elementary integral. This means that modern realization of so-called 'Risch algorithm' are not able to work whit algebraic functions.

Remark

Maple sometimes gives very strange results (errors?).

```
> u := simplify(diff(ln(x+sqrt(x^3+x+1)), x))
```

$$u := \frac{3x^2 + 2\sqrt{x^3+x+1} + 1}{\sqrt{x^3+x+1}\left(2x + 2\sqrt{x^3+x+1}\right)}$$

```
> int(u, x)
```

$$\frac{\left(\sum_{\_R = RootOf(\_Z^3 - \_Z^2 + \_Z + 1)} \frac{\ln(x - \_R)}{3\_R^2 - 2\_R + 1}\right)}{2} + \frac{1}{9504}\left(I\sqrt{6}\, 12^{1/3} \left(\sum_{\_\alpha = RootOf(\_Z^3 - \_Z^2 + \_Z +$$

Abelian integrals

It is well-known, that abelian integral is an integral of the form

$$\int R(x, y) dx,$$

here R is an arbitrary rational function of the two variables x and y related by the equation

$$f(x, y) = 0,$$

where f is an irreducible polynomial from $\mathbb{Q}[x, y]$.
Theorem about the local uniformisation tell us that the arc of the curve at neighborhood of a point (a, b) can be described by means of power series

$$x = x_t = a + c_1 t^{n_1} + \dots, \quad y = y_t = b + b_1 t^{m_1} + \dots.$$

This circumstance allows to enter a concept of an order of a singularity.

Hauptfunktion

Definition

If (x', y') is a point on this curve then there is such function $H \in \mathbb{C}(x, y)$ that (x', y') is a simple pole of H and the residue at this point is equal 1. Such function with minimal order $r = 1 + p$ is called a fundamental function (Hauptfunktion) and the number p is called a genus (Rang) of curve.

Trivial statement of the existence of the fundamental function is the unique existence theorem in Weierstass lectures. Here we can find the algorithm for calculation of Hauptfunktion.

Hauptfunktion for elliptic curve

In many cases we can write expression for fundamental function explicitly, so for elliptic curve

$$y^2 = a_0 y^3 + a_1 y^2 + a_2 y + a_3$$

the fundamental function is equal

$$\frac{1}{2y'} \frac{y + y'}{x - x'}.$$

This function has pole at (x', y') and at infinity, the genus is equal to 1.

Integrals of 3 kinds

Let $(a_1, b_1), \ldots, (a_p, b_p)$ be poles of the fundamental function with respect to first argument. Coefficients of Laurent series

$$H(x_t, y_t; x', y')dx' = H_n(x', y')dx' \cdot \frac{1}{t} + c_0(x', y') - H_n'(x', y')dx' \cdot t + \ldots$$

at (a_n, b_n) give us abelian integrals of the 1st and the 2nd kinds. Expression in Laurent series with respect to second argument has the form

$$H(x, y; x_t, y_t)\frac{dx_t}{dt} = \frac{\delta}{t} + c_0 + c_1 t + \ldots;$$

where the residue $\delta \neq 0$ iff center of the arc (x_t, y_t) coincides with (x, y) or yet one singular point. So the expression $H(x, y; x', y')dx'$ with respect to second argument is known in other theories as an integral of the 3rd kind.

Decomposition of abelian integral into integrals of 3 kinds

For any rational function R we can write abelian integral

$$\int R(x,y)dx$$

as sum of algebraic part $R'(x,y)$, log-part

$$\sum_m c_m H(x_m, y_m; x, y)dx$$

with log-singularities in poles of R and the 3rd part

$$\sum_{n=1}^{p} g_n' \int H_n(x,y)dx - g_n \int H_n'(x,y)dx$$

with simple poles in fixed singularities $(a_1, b_1), \dots, (a_p, b_p)$ of the fundamental function.

What for?

In lectures of Weierstrass there are explicit expressions (!) for the coefficients of the decomposition and for the algebraic part.
However there are not any explanation about usage this excellent formula. I believe that:

- the equations $g_1 = \cdots = g'_p = 0$ and all $c_m = 0$ are necessary and sufficient conditions for integration in algebraic functions,
- the p equations $g_1 = \cdots = g_p = 0$ are necessary conditions for integration in elementary functions,
- the existence of rational functions with poles in log-singularities of integral give us the sufficient conditions for integration in elementary functions (the accurate formulation demands more place)

If given integral is not elementary then the decomposition reduces this integral to standard integral with well-known properties.

The end.

Calculations made in **??**.

See additional materials on `http://malykhmd.neocities.org`.