On periodic approximate solutions of the three-body problem found by conservative difference schemes

Report for CASC'2020

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Abstract

We will discuss the possibility of using implicit difference schemes to study the qualitative properties of solutions of dynamical systems, primarily the periodicity of the solution.
Here we formulate the Lagrange problem of finding all *approximate* solutions of the three-body problem at which the distances between the bodies remain constant. The proved two theorems reduce the problem to the study of the properties of linear oscillators. In particular, it turns out that the periodicity of the exact solution in the Lagrange case, when the bodies form a regular triangle, is inherited by the approximate solution.

What we want?

- The standard numerical method for studying dynamical systems suggests replacing derivatives in differential equations with finite differences and thus reduces the numerical analysis of a dynamical system to solving systems of algebraic equations.
- We believe that the approach that provides a simple and effective tool for the approximate calculation of solution parameters can be no less effective in a qualitative analysis.
- The ultimate goal of our research is to clarify how to use the finite difference method for a qualitative analysis of dynamical systems.

Dynamical system (D) and finite-difference scheme (S)

Consider a dynamical system (D) of differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n), \quad i = 1, 2 \ldots, n,$$

where $x = (x_1, \ldots x_n)$, and f is a rational function of its argument.
Any finite-difference scheme for this system is a system (S) of algebraic equations

$$g_i(x, \hat{x}, dt) = 0, \quad i = 1, 2 \ldots, m,$$

that determine the relation between the value of the variable x, taken at a certain moment of time t, and the value $\hat{x}$ of this variable taken at the moment of time $t + dt$.
In this case $x, \hat{x}$ can be considered as two lists of symbolic variables $[x_1, \ldots, x_n]$ and $[\hat{x}_1, \ldots, \hat{x}_n]$. Even the step dt can be treated as a symbolic variable.

Approximate solutions

Let us take algebraic numbers for the coordinate of the point x_0 and the step dt. We will understand the approximate solution of the initial-value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0,$$

found according to the finite-difference scheme (S) as a finite or infinite sequence

$$\{x_0, x_1, x_2, \dots\},$$

whose elements are determined recursively. To find x_{n+1}, one has to do the following:

- substitute $x = x_n$ into the system (S),
- find the solution $\hat{x}$ of this system that tends to x_n when $dt \to 0$, and accept this solution for x_{n+1}.

In this treatment, we will interpret x_n as approximate solutions of the initial-value problem at point $t = ndt$.

Main idea

Traditionally, the focus of numerical analysis is on the accuracy of approximation, the proximity of an approximate and exact solution, and its stability with respect to rounding errors.
The finite difference method produces algebraization of the problem, that is, it brings the problem to the form most convenient for using purely algebraic tools like CAS.
For inheritance by an approximate solution of the properties of an exact solution, it is extremely important that the approximate solution is found using conservative schemes, i.e., schemes that preserve all algebraic integrals of motion.

The simplest example: preserving of an integral

Consider a linear oscillator

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x,$$

There is one algebraic integral

$$x^2 + y^2 = C,$$

The standard Euler scheme does not preserve it, while the midpoint scheme

$$\hat{x} - x = -(\hat{y} + y)\frac{dt}{2}, \quad \hat{y} - y = (\hat{x} + x)\frac{dt}{2}$$

preserves it according to Cooper theorem.

The simplest example: inheritance of periodicity

If we take for α the minimal positive root of the equation

$$(1+i\alpha)^{2N} = (1+\alpha^2)^N,$$

which in terms of trigonometric functions can be expressed as

$$\alpha = \tan\frac{\pi}{N},$$

then the calculation according to the midpoint finite-difference scheme with the step

$$dt = 2\alpha = 2\tan\frac{\pi}{N},$$

in N steps leads to the initial values of x, y.
Ref.: Gerdt V.P. et al. // Discrete and Continuous Models and Applied Computational Science 27(3), 242–262 (2019).

Conservative schemes for the problem of many bodies

The classical problem of n bodies:

$$m_i \ddot{\vec{r}}_i = \sum_{j=1,\, j\neq i}^{n} \gamma \frac{m_i m_j}{r_{ij}^3} \left(\vec{r}_j - \vec{r}_i\right), \quad i = 1, \ldots, n$$

- 1992. D. Greenspan proposed the first finite-difference scheme for the many-body problem, preserving all classical integrals of motion.
- 2019. Hong Zhang et al. proposed invariant energy quadratization method (IEQ method) when we "transform the energy into a quadratic form of a new variable via a change of variables"

Our conservative scheme for the problem of many bodies

We introduce the following additional variables

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}, \quad \rho_{ij} = \frac{1}{r_{ij}}$$

and arrive at a system of ordinary differential equations with rational right parts.

All algebraic integrals of a many-body problem are quadratic integrals of this system. The additional integrals

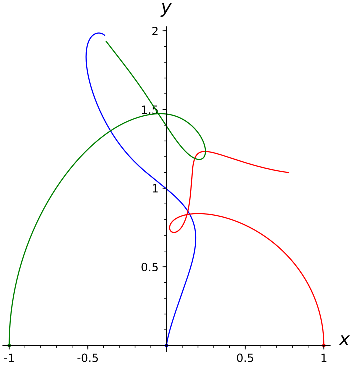
$$r_{ij}^2 - (x_i - x_j)^2 - (y_i - y_j)^2 - (z_i - z_j)^2 = \text{const}, \quad r_{ij}\rho_{ij} = \text{const}, \quad i \neq j,$$

are also quadratic.

According to the Cooper theorem, the midpoint scheme preserves all quadratic integrals of motion exactly, therefore, it is a conservative scheme for the many-body problem.

Numerical experiments

We made several numerical experiments with flat three body problem using suggested scheme and noticed that, in the case of Lagrange, distances are saved at the rounding error level.
Thus we decided to investigate this case pure algebraical methods without any rounding error.



Ref: Gerdt V.P. et al. // ArXiv. 2007.01170.

Discrete analogue of Lagrange problem

To investigate the inheritance by the approximate solution of the properties of the exact solution in Lagrange case, we consider the following problem.

Problem

Using the midpoint scheme, find all approximate solutions of the planar three-body problem, in which the distances between the bodies are unchanged

$$\hat{r}_{ij} = r_{ij}, \quad i \neq j, \tag{1}$$

and the constraint integrals have their natural values

$$r_{ij}^2 - (x_i - x_j)^2 - (y_i - y_j)^2 = 0, \quad r_{ij}\rho_{ij} = 1. \tag{2}$$

First attempt to solve

To solve the problem we have to investigate the projections of a certain algebraic set M in the 19-dimensional affine space. In theory, this problem is pure algebraic and algorithmically solvable. We have tried to solve it in CAS Sage using standard instruments for working in polynomial rings. We obtained an ideal J in the ring

$$\mathbb{Q}[a, b, c, dt, x_1, x_2, x_3, y_1, y_2, y_3, u_1, u_2, u_3, v_1, v_2, v_3].$$

Sage is unable to answer even such trivial questions about this ideal as belonging of element $y_1 + y_2 + y_3$ to ideal J:

```
sage: J=lagrange_ideal()
sage: K(y1 + y2 + y3) in J
```

Probably, the number of variables is still too large for calculations on a common computer. Therefore, the problem will have to be solved partially by hand.

Second attempt to solve: two theorems

- If Lagrange problem allows a solution, then the projections of the coordinates and velocities on the axes Ox and Oy can be found as solutions of a linear problem

$$\begin{cases} \dot{z}_i = w_i, \quad i = 1, 2, 3 \\ m_i \dot{w}_i = -\dfrac{\partial U}{\partial z_i}, \quad i = 1, 2, 3, \end{cases} \tag{3}$$

found using the midpoint scheme.

- Let a pair of solutions $\{z_i = x_i, w_i = u_i\}$ and $\{z_i = y_i, w_i = v_i\}$ to the problem (3) is found using the midpoint scheme and is constrained by three equations

$$(x_i - x_j)^2 + (y_i - y_j)^2 = a_{ij}^2, \quad i \neq j,$$

Then it may be raised to an approximate solution of the Lagrange problem, accepting

$$r_{ij} = a_{ij}, \quad \rho_{ij} = 1/a_{ij}.$$

Second attempt to solve: existence of periodical solution

One class of solutions traditionally associated with the name of Lagrange, can be obtained by assuming that all the distances a_{ij} are equal to one constant a. We investigated the system (3) for this case in Sage successfully.

Corollary

There exists a family of approximate solutions to the three-body problem, on which the bodies form a regular triangle with the constant side a.This solution can be raised from two solutions of a linear dynamical system (3) *found using the midpoint scheme.*

As we saw above, the step dt in the midpoint scheme for the system (3) can be chosen in such a way that the approximate solution becomes periodic, and its frequency tends to the frequency of the exact solution.

What we need?

At the moment, our experiments with the three-body problem are limited not so much by present-day computer capacities, which are still not enough to work with dozens of symbol variables, but rather by very incomplete kit of tools implemented to exclude unknowns. The midpoint scheme for the many-body problem is rich in discrete symmetries. The exclusion method implemented in Sage is based on the Gröbner bases with lex ordering and does not take the symmetry of the system into account, but sometimes the usage of such symmetries speeds up computing [Steidel,Faugere].
We believe that the 'factorization' of the system by these symmetries can reduce the number of unknowns by n times, where n is the number of bodies in the system.

The end.

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