Dynamical systems with a quadratic right-hand side

PCA'2021

Ali Baddour, Oleg Kroytor, **Mikhail Malykh**, and Leonid Sevastianov

Peoples' Friendship University of Russia, Moscow, Russia

Apr. 19-23, 2021, ver. April 18, 2021

This work is supported by the Russian Science Foundation (grant no. 20-11-20257).

Dynamic systems

Let us consider dynamical system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_m), \quad i = 1, 2, \ldots, m. \tag{1}$$

with polynomial right-hand side

$$f_i \in \mathbb{Q}[x_1, \ldots, x_m].$$

For brevity, we will use vector notation, meaning by x the tuple $(x_1, \ldots, x_m)$.

Finite difference method

Within the framework of the finite difference method the system of differential equations is replaced with the system of algebraic equations

$$g_i(x, \hat{x}, \Delta t) = 0, \quad i = 1, \dots, n. \tag{2}$$

In this case, x is interpreted as the value of the solution at the time t, and $\hat{x}$ as the solution at the time $t + \Delta t$.

Example

- Euler scheme $\hat{x} - x = f(x)\Delta t$,
- midpoint scheme $\hat{x} - x = f\left(\frac{\hat{x}+x}{2}\right)\Delta t$,
- trapezoid scheme $\hat{x} - x = \frac{f(\hat{x})+f(x)}{2}\Delta t$, ...

t-symmetry

From the point of view of mathematical modeling of mechanical phenomena, the inheritance of two properties by the difference scheme, namely, t-symmetry and reversibility, is of greatest importance.

Definition

We say that the difference scheme has t-symmetry if it is invariant under the transformation

$$\Delta t \to -\Delta t, \quad x \to \hat{x}, \quad \hat{x} \to x.$$

Example

- Euler scheme $\hat{x} - x = f(x)\Delta t$ is not t-symmetric,
- midpoint scheme $\hat{x} - x = f\left(\frac{\hat{x}+x}{2}\right)\Delta t$ is t-symmetric,
- trapezoid scheme $\hat{x} - x = \frac{f(\hat{x})+f(x)}{2}\Delta t$ is t-symmetric also.

Reversibility

Definition

By reversibility, we should understand the possibility to uniquely determine the final data $\hat{x}$ from the initial data x and vice versa using the system

$$g_i(x, \hat{x}, \Delta t) = 0, \quad i = 1, \ldots, n,$$

for any fixed value of the step Δt.

Since Eqs. (2) are algebraic, this means that $\hat{x}$ must be a rational function of x, and x must be a rational function of $\hat{x}$.

Example

Euler scheme $\hat{x} - x = f(x)\Delta t$ written for linear dynamical system is reversible.

Cremona transformations

We will consider $x, \hat{x}$ as two points of the projective space $\mathbb{P}_m$ and say that the difference scheme

$$g_i(x, \hat{x}, \Delta t) = 0, \quad i = 1, \dots, n,$$

is invertible if for any fixed value of Δt this scheme defines a Cremona transformation.
The combination of t-symmetry and reversibility means that the difference scheme defines a one-parameter family of Cremona transformations $\mathcal{C}$, such that

$$\hat{x} = \mathcal{C}(\Delta t)x$$

and

$$\mathcal{C}(\Delta t)^{-1} = \mathcal{C}(-\Delta t).$$

Reversibility in mechanics

Let us consider the following Painlevé initial problem

$$\frac{dx}{dt} = f(x), \quad x|_{t=t_0} = x_0 \tag{3}$$

on the segment $[t_0, t_0 + \Delta t]$ of the real axis t.

For some values of t_0, the procedure for analytic continuation of the solution obtained in the Cauchy theorem along a segment does not encounter singular points other than poles, and in this case the final value of $x(t_0 + \Delta t)$ is uniquely determined by the initial value of x_0. However, if the path encounters a branch point, then the final value depends on the way it is passed. Therefore, $x(t_0 + \Delta t)$ is a multivalued function of the initial value x_0.

Painlevé property

If a dynamical system has the global reversibility property, then it also has the Painlevé property.

Definition

A dynamical system has the Painlevé property, if the singular points of the solution are not branch points.

Example

Classical completely integrable models, including pendulums and tops, are integrable in elliptic functions and, as can be seen from the solution, have the Painlevé property.

One-dimensional case

In the one-dimensional case ($n = 1$), only the Riccati equation has the Painlevé property

$$\frac{dx}{dt} = a + bx + cx^2 \tag{4}$$

for any, including zero values of constants a, b, c. Moreover, the initial problem defines a Möbius transformation on the projective line.

It is not difficult to construct a difference scheme that inherits this property:

$$\hat{x} - x = \left(a + b\frac{x + \hat{x}}{2} + cx\hat{x} \right) \Delta t \tag{5}$$

Riccati equation

Since any birational transformation on a projective line is a Möbius one, it is easy to prove the converse.

Theorem (Malikh M.D., 2019)

In the one-dimensional case, an invertible difference scheme can be constructed only for the Riccati equation

Ref.: E. A. Ayryan et al. On Difference Schemes Approximating First-Order Differential Equations and Defining a Projective Correspondence Between Layers. Journal of Mathematical Sciences 240 (2019), 634–645. DOI: 10.1007/s10958-019-04380-0

Many dimensional case

For $n > 1$, the continuous and discrete case lose their similarity.

1. For any dynamical system with a quadratic right-hand side, a t-symmetric reversible difference scheme can be constructed:

$$\hat{x}_i - x_i = F_i(x, \hat{x})\Delta t, \quad i = 1, \ldots, n, \tag{6}$$

   where F_i is obtained from f_i by replacing monomials: x_j with $(\hat{x}_j + x_j)/2$, $x_j x_k$ with $(\hat{x}_j + x_j)(\hat{x}_k + x_k)/4$, and x_j^2 with $x_j \hat{x}_j$.
2. Only a few dynamical systems with a quadratic right-hand side possess the Painlevé property.

Example

The dynamical system describing the rotation of a rigid body around a fixed point always has a quadratic right-hand side and has the Painlevé property only in 4 special cases found by S.V. Kovalevskaya.

$\wp$-oscillator

Dynamic system

$$\dot{x} = y, \quad \dot{y} = 6x^2 - a \tag{7}$$

has an algebraic integral

$$\frac{y^2}{2} - 4x^3 + ax = C_1, \tag{8}$$

the meaning of which is the total mechanical energy. If the value of C_1 is fixed, then the general solution

$$x = \wp(t + C_2, 2a, C_1), \quad y = \wp'(t + C_2, 2a, C_1), \tag{9}$$

defines a birational correspondence between the initial point (x_0, y_0) and the final point (x_1, y_1) on the integral elliptic curve (8). This correspondence does not extend to a birational transformation of the projective plane xy.

$\wp$-oscillator, 2

Nevertheless, we can approximate the system

$$\dot{x} = y, \quad \dot{y} = 6x^2 - a$$

by the difference scheme

$$\hat{x} - x = \frac{\hat{y} + y}{2}\Delta t, \quad \hat{y} - y = (6x\hat{x} - a)\Delta t \tag{10}$$

which defines a birational correspondence between the points (x, y) and $(\hat{x}, \hat{y})$ of the projective plane. In this case, the integral of motion

$$\frac{y^2}{2} - 4x^3 + ax = C_1$$

is not preserved. Thus, in the transition from a continuous model to a discrete one, the algebraic integral of motion is not preserved and thanks to this fact we move from the group of birational transformations of an elliptic curve to the group of Cremona transformations.

Inherit of Painlevé property

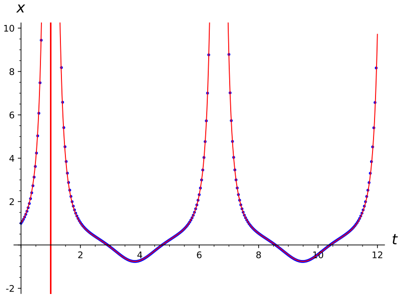
We investigated how the Painlevé property is inherited by an approximate solution using two examples, the Riccati equation and the $\wp$-oscillator.
In both cases, it turned out that the calculations using a reversible difference scheme can be continued through a pole without noticeable accumulation of errors.

Inherit of Painlevé property for the $\wp$-oscillator

Figure shows the solution of system

$$\dot{x} = y, \quad \dot{y} = 6x^2 - \frac{1}{2}$$

with initial conditions $(x, y) = (1, 2)$. The calculation of an approximate solution according to a reversible scheme does not encounter any difficulties over the entire considered interval $0 < t < 12$, containing two poles of the exact solution.



Periodicity of approximate solutions

The approximate solution is a sequence $x_0, x_1, \dots,$ each next element of which is obtained from the previous one by applying the Cremona transformation $\mathcal{C}$:

$$x_{x+1} = \mathcal{C} x_n$$

This sequence will have period n, if $x_n = x_0$, i.e., if x_0 is a fixed point of $\mathcal{C}^n$.

Calculation of the step when a period and an initial value are given

Problem

Let a positive integer n and an initial value $x_0 \in \mathbb{Q}^m$ be given. We want to calculate the step Δt at which the sequence has a period n.

Considering Δt as a symbolic variable, we calculate $\mathcal{C}^n x_0$. We get m rational functions from $\mathbb{Q}(dt)$. Equating them to x_0, we obtain m of algebraic equations, the common roots of which are the required step values.
Generally speaking, several equations for one variable may not have common roots, but they have common roots in all our examples.

Examples

We have considered three examples:

1. a linear oscillator that can be easily investigated analytically,
2. $\wp$-oscillator, and
3. Jacobi oscillator, i.e., dynamical system

$$\dot{p} = qr,\ \dot{q} = -pr,\ \dot{r} = -k^2pq, \tag{11}$$

integrable in terms of elliptic Jacobi functions.

We chose different initial data and considered n in the interval from 2 to 10. The degrees of polynomials, the common roots of which give the desired step values, increase exponentially with n, which significantly limited our ability to increase n.

Example 2. $\wp$-oscillator with $a = \frac{1}{2}$

For the $\wp$-oscillator with $a = \frac{1}{2}$ under the same initial conditions as were used in the example above, there are no values of step Δt for which the solution has a period $n = 2, 3, 6$. For $n = 4$ the step is independent of the starting point. The table contains all the matched positive values for Δt found for n in the top ten.

n	Δt
2	$\emptyset$
3	$\emptyset$
4	1.074
5	6.908
6	$\emptyset$
7	$0.556, 5.870, 7.759$
8	$0.535, 1.074, 6.843$
9	$0.504, 9.187$
10	$0.471, 0.559, 6.777, 6.908$

Example 3. Jacobi oscillator with $k = \frac{1}{5}$

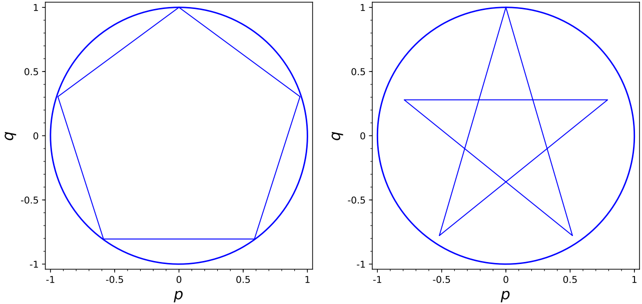
For the Jacobi oscillator with $k = \frac{1}{5}$ under the initial conditions

$$p = 0, \quad q = 1, \quad r = 0,$$

there are positive values of step Δt for any periods $n > 2$.

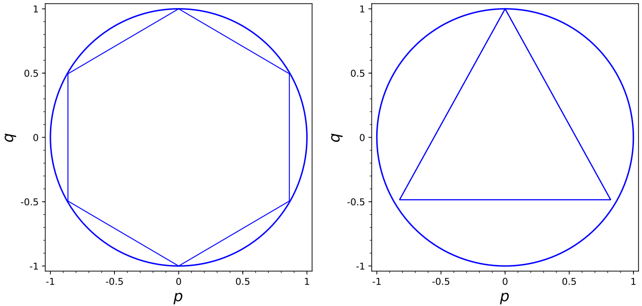
n	Δt
2	$\emptyset$
3	3.609
4	2.041
5	$1.47,\ 6.86$
6	$1.17,\ 3.60$
7	$0.97,\ 2.57,\ 10.85$
8	$0.83,\ 2.04,\ 5.18$
9	$0.73,\ 1.70,\ 3.60,\ 16.23$

Example 3. Jacobi oscillator, $n = 5$



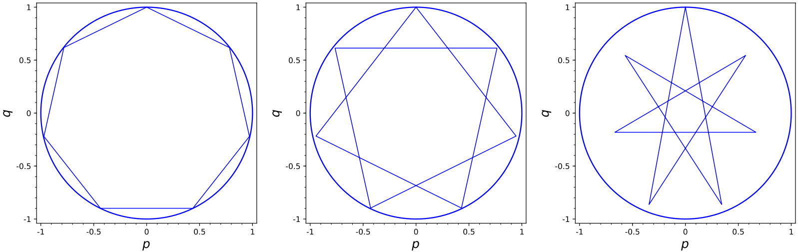
Approximate solution has the period $n = 5$ at two values of the step. In the plane pq, at the first value of the step, an almost regular pentagon is obtained and at the second value we obtain a pentagram.

Example 3. Jacobi oscillator, $n = 6$



Approximate solution has the period $n = 6$ at two values of the step. First of them is coincide with the step at $n = 3$ and give us a triangle, the second give a hexagon.

Example 3. Jacobi oscillator, $n = 7$



Approximate solution has the period $n = 7$ at three values of the step.
In all cases, the integral of motion $p^2 + q^2 = 1$ is not exactly conserved.

Example 3. Jacobi oscillator, $n \to \infty$

As n grows, the number of step values at which periodic approximations to the solution of the Cauchy problem are obtained grows. The smallest possible Δt for fixed n corresponds to an almost regular n-gon in the pq plane.
These solutions revert to their original value in times $n\Delta t$, collected in Table. These times seem to form a monotonically decreasing sequence converging to the exact period.

n	$n\Delta t$
3	10.827
4	8.164
5	7.379
6	7.022
7	6.827
8	6.706
9	6.627
∞	6.347

Results of computer experiments

It is convenient to present the results of the experiments carried out in the form of two hypotheses:

1. for any sufficiently large n and any initial conditions, one can specify a finite number of positive values for the step Δt, at which periodic sequences with the period n are obtained,
2. if we associate each n with a minimum period, we get a sequence converging to the period of the exact solution for $n \to \infty$.

By virtue of the first hypothesis, any exact particular solution can be approximated by an approximate solution that inherits the periodic nature of the exact solution, and by virtue of the second hypothesis the approximation step Δt can be taken arbitrarily small and, therefore, approach the exact solution with any given accuracy.

Equiperiodic sets

In the previous Section, we followed one solution, but changed Δt. Let us now look at the behavior of solutions in the phase space, but for a fixed Δt.

Definition

The set in the phase space formed by all the initial data generating approximate solutions with the same period n is algebraic; we will call it an equiperiodic set of the n-th order.

The equiperiodic set E is an invariant set for difference model:

$$x \in E \Rightarrow \hat{x} \in E.$$

It is easy to deduce from the first hypothesis that equiperiodic sets of sufficiently large order are not empty and have codimension 1.

Calculation of equiperiodic sets

Problem

Let a positive integer n and a step Δt be given. We want to calculate the equations described the equiperiodic set of the n-th order.

To find it in the previous algorithm, it is necessary to consider x_0 as a tuple of m symbolic variables. We managed to find these sets only for small n.

Example. $\wp$-oscillator with $a = \frac{1}{2}$

- At $n = 2$ and 3 the equiperiodic sets are empty.
- At $n = 4$ the curve equation degenerates into

$$3\Delta t^4 - 4 = 0,$$

  thus the step is independent of the initial data.
- At $n = 5$ the equiperiodic set appears to be an elliptic curve

$$27\Delta t^{10} x - 432\Delta t^8 x y^2 + 432\Delta t^8 x^2 + 1728\Delta t^6 x^3 + 27\Delta t^8$$
$$-432\Delta t^6 y^2 - 936\Delta t^6 x + 168 dt^4 + 240\Delta t^2 x - 80 = 0.$$

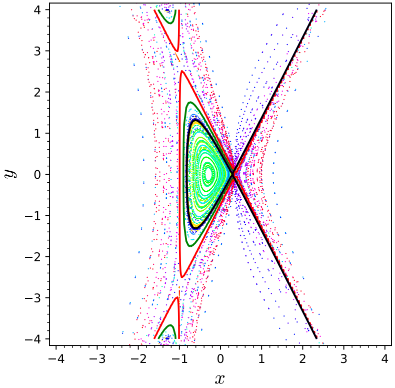
Degrees of the equiperiodic curves

The degrees of $F_n \in \mathbb{Q}[\Delta t][x, y]$ are presented in the table.
Due to the degree grows , the equiperiodic curves do not belong to the same sheaf, linear or irrational.

n	Degre of F_n
4	0
5	3
6	3
7	6
8	6
9	9
10	12

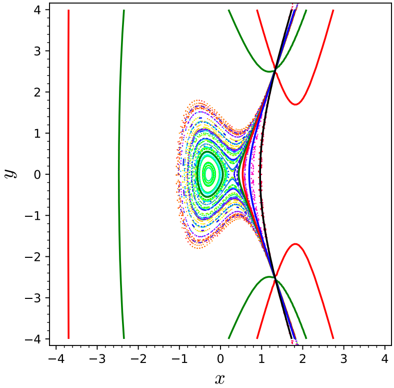
$\wp$-oscillator. Phase diagram, $\Delta t = 1$

Points of approximate solutions generated by the initial values belonging to the square $[-1,1] \times [-1,1]$ with space step $d = 1/5$, colored in HUE, and equiperiodic curves (F_5 in black, F_6 in red, F_7 in blue, F_8 in green).



$\wp$-oscillator. Phase diagram, $\Delta t = \frac{1}{2}$

Points of approximate solutions generated by the initial values belonging to the square $[-1, 1] \times [-1, 1]$ with space step $d = 1/5$, colored in HUE, and equiperiodic curves (F_5 in black, F_6 in red, F_7 in blue, F_8 in green).
It is well seen that most of the diagram points group around these curves.



$\wp$-oscillator. Phase diagram, $\Delta t = 0.471\ldots$

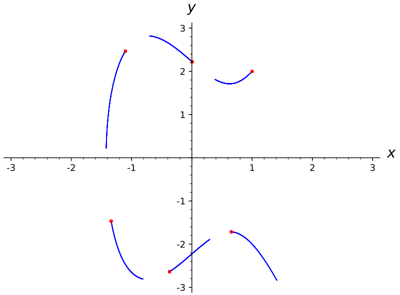
When $\Delta t = 0.471\ldots$, the approximate solution of the initial problem with the condition

$$x = 1, \quad y = 2$$

has the period $n = 10$. Thus the trajectory on xy plane consists from 10 points.
If we perturb the initial condition and take

$$x = 1.0001, \quad y = 2,$$

then the trajectory is infinite set of the points.



These points lie on a curve which repeat the form of F_{10}. Is this curve algebraic?

Differential vs. difference models

Differential model

$$\dot{x} = f(x)$$

has

1. periodic solutions $x(t+T) = x(t)$,
2. algebraic integral, i.e. there is a linear invariant sheaf

$$g(x) + ch(x) = 0,$$

3. the model define a birational transformation on integral manifold

Difference model

$$F(\hat{x}, x, \Delta t) = 0$$

with fixed step Δt has

1. sequence of equiperiodic sets $F_3, F_4, \dots,$
2. they are invariant sets for dynamical system, but they don't form a linear or irrational sheaf.
3. the model define Cremona transformation.

Invertibility vs. exact preservation of all algebraic integrals

The our ultimate goal is to create discrete models that have the most important properties of mechanical models. These include undoubtedly the inheritance of algebraic conservation laws, t-symmetry, reversibility and periodicity.
It is impossible to combine reversibility and exact preservation of all algebraic integrals.

Ref. E. A. Ayryan et al. On Explicit Difference Schemes for Autonomous Systems of Differential Equations on Manifolds. Lecture Notes in Computer Science 11661 (2019), 343–361. DOI: 10.1007/978- 3-030-26831-2_23.

Cremona transformation vs. birational transformation on an integral manifold

The starting point for this article was the observation that dynamical systems with a quadratic right-hand side can be approximated by reversible difference schemes with t-symmetry. The approximate solutions found using these schemes are birational functions of the initial data over the entire phase space. This is surprising since in the continuous case, for this property to appear, one had to restrict the phase space by an algebraic integral manifold.

Historical remark

Hermite believed that the theory of birational transformations of curves would be a section in the theory of the Cremona group, which F. Klein considered an annoying mistake, which he considered necessary to describe in detail in his Lectures on the History of Mathematics.
Now it turns out that Hermite was right after all, and there is a connection between birational transformations on curves and Cremona transformations, which manifests itself in the discretization of dynamical systems.

The End

Additional materials: `https://malykhmd.neocities.org`